Homework 2

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Problem 1

Assume we are given 7 distinct points in the plane $\mathbb{R}^2$ with the following property: any line that goes through two of the points is going through at least three of the points. Prove that all 7 points have to lie on one line.

Assume we can construct a convex hull around the set of points. If we cannot construct a convex hull around it, this means that the set is colinear. We must also follow the property that any line going through two of the points will go through at least three of the points. The convex hull is formed by connecting edges between points in the set, which means that every edge must have three points. This means that at least 8 points are needed to form a four-sided convex hull. There are not enough points to do so, so the convex hull must be three-sided. Then there are three points as vertices and three points on the edges. The question now is where to place the seventh point. Suppose it is on one of the edges. Then consider the point at the opposite vertex. In order to satisfy the property, we need to add another point along the line from the edge point to the opposite vertex, but this contradicts the fact that we can only have 7 points. Now suppose instead that it is inside the convex hull. Then it is on a line either from an edge point to an edge point or from an edge point to a vertex point such as to preserve the property. If the seventh point is on the line from an edge point on the line AB to an edge point on the line AC, then consider the line from the seventh point to the vertex point B (C works too). There are only two points on this line, the vertex point B and the seventh point. It cannot intersect the vertex A because then the seventh point would be on the hull edge AB, which contradicts the premises. It cannot intersect the edge on the line AC because then the seventh point would need to be on the hull edge AC, which also contradicts the premises that the seventh point is in the interior of the hull. Now consider that the seventh point is on the line from an edge point AB to a vertex point C. Then either it is the case that the line from the vertex A to the seventh point has only two points or the line from the point on the line AC to the seventh point. If both were true, then the seventh point would be colinear with A and AC, which contradicts the premises that the seventh point is in the interior of the hull. Therefore, overall, no matter where we put the seventh point, it fails to follow the given property. As such, we cannot contruct a valid convex hull around this set. Therefore, all points are colinear to each other.

Problem 2

Show that for all $n \in \mathbb{N}$, the nth Fibonacci number F_n satisfies

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

The nth Fibonacci number is defined as follows:

$$F_n = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ F_{n-2} + F_{n-1} & \text{else} \end{cases}$$

Our inductive proposition $P(q)$ is that the qth Fibonacci number is equal to the closed formula provided above. $P(1) = 1$ and $P(2) = 1$, as expected. Therefore, the base case holds. In our inductive step, we will show with strong induction that $P(q-1)$ and $P(q)$ imply $P(q+1)$. For simplicity of notation, let $k = p - 1$. The kth Fibonacci number is defined by the sum of the $k-2$th and $k-1$th Fibonacci numbers. Using the inductive proposition, this is equal to

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k-2}}{\sqrt{5}}+\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k-2}+\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$=\frac{\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}+\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\right)-\left(\left(\frac{1-\sqrt{5}}{2}\right)^{k-2}+\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right)}{\sqrt{5}}$$

$$=\frac{\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}+\left(\frac{1+\sqrt{5}}{2}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}\right)-\left(\left(\frac{1-\sqrt{5}}{2}\right)^{k-2}+\left(\frac{1-\sqrt{5}}{2}\right)\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{k-2}\right)}{\sqrt{5}}$$

$$=\frac{\left(\left(\frac{1+\sqrt{5}}{2}+1\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}\right)-\left(\left(\frac{1-\sqrt{5}}{2}+1\right)\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{k-2}\right)}{\sqrt{5}}$$

$$=\frac{\left(\left(\frac{2+2\sqrt{5}}{4}+1\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}\right)-\left(\left(\frac{2-2\sqrt{5}}{4}+1\right)\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{k-2}\right)}{\sqrt{5}}$$

$$=\frac{\left(\left(\frac{2+2\sqrt{5}+4}{4}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}\right)-\left(\left(\frac{2-2\sqrt{5}+4}{4}\right)\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{k-2}\right)}{\sqrt{5}}$$

$$=\frac{\left(\left(\frac{1+2\sqrt{5}+5}{4}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}\right)-\left(\left(\frac{1-2\sqrt{5}+5}{4}\right)\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{k-2}\right)}{\sqrt{5}}$$

$$=\frac{\left(\left(\frac{1+\sqrt{5}}{2}\right)^{2}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}\right)-\left(\left(\frac{1-\sqrt{5}}{2}\right)^{2}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{k-2}\right)}{\sqrt{5}}$$

$$=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k}-\left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}}$$

Substituting $k=q+1$ gives us $\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{q+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{q+1}}{\sqrt{5}}$, or $P(q+1)$. Therefore, by the principle of induction, $P(q)$ holds for all $q\in\mathbb{Z}$.

Problem 3

Give an example of an infinite collection $S_1, S_2, ...$ of closed sets whose union $\cup_{i=1}^{\infty} S_i$ is not closed.

Let S_k be the closed set $[1/2^k, 1]$. Then $\cup_{i=1}^{\infty} S_i$ is the open set $(0, 1]$, with supremum 1 and infimum 0.

Problem 4

Problem 4a

Let $w_1 = (1, 0)$, $w_2 = (-1/2, \sqrt{3}/2)$, and $w_3 = (-1/2, -\sqrt{3}/2)$. Show that

$$\forall x \in \mathbb{R}^2 : \frac{3}{2}\|x\|^2 = \langle x, w_1\rangle^2 + \langle x, w_2\rangle^2 + \langle x, w_3\rangle^2$$

Let x_1, x_2 denote the first and second components of x. Then we can expand the equality as follows:

$$\begin{aligned}
\frac{3}{2}(x_1^2 + x_2^2) &= \langle x, w_1\rangle^2 + \langle x, w_2\rangle^2 + \langle x, w_3\rangle^2 \\
&= (x_1)^2 + \left(-\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2\right)^2 + \left(-\frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2\right)^2 \\
&= x_1^2 + \left(\frac{x_1^2 - 2\sqrt{3}x_1x_2 + 3x_2^2}{4}\right) + \left(\frac{x_1^2 + 2\sqrt{3}x_1x_2 + 3x_2^2}{4}\right) \\
&= x_1^2 + \frac{2x_1^2 + 6x_2^2}{4} \\
&= \frac{3}{2}x_1^2 + \frac{3}{2}x_2^2 \\
&= \frac{3}{2}(x_1^2 + x_2^2)
\end{aligned}$$

Problem 4b

Find a set of 5 vectors $w_1, w_2, ..., w_5$ such that

$$\forall x \in \mathbb{R}^2 : \frac{5}{2}\|x\|^2 = \sum_{i=1}^{5}\langle x, w_i\rangle^2$$

Let $w_1 = (1, 0)$, $w_2 = \left(\cos\left(\frac{2\pi}{5}\right), \sin\left(\frac{2\pi}{5}\right)\right)$, $w_3 = \left(\cos\left(\frac{4\pi}{5}\right), \sin\left(\frac{4\pi}{5}\right)\right)$, $w_4 = \left(\cos\left(\frac{6\pi}{5}\right), \sin\left(\frac{6\pi}{5}\right)\right)$, $w_5 = \left(\cos\left(\frac{8\pi}{5}\right), \sin\left(\frac{8\pi}{5}\right)\right)$. These correspond to the 5th roots of unity. Indeed, upon manual evaluation, $\sum_{q=0}^{4}\left(\cos\left(\frac{2q\pi}{5}\right)x_1 + \sin\left(\frac{2q\pi}{5}\right)x_2\right)^2 = \frac{5x_1^2+5x_2^2}{2} = \frac{5}{2}\|x\|$.

Problem 5

> Let $f(x) = 1/q$ if $x = p/q$, where p, q are integers with no common factors, and $f(x) = 0$ if x is irrational. Find the points at which $f(x)$ is continuous.

We will show that $f(x)$ is continuous for all irrational values of x. Let a be some irrational number, and ϵ be some arbitrary real larger than zero. If a is negative, we can redefine a as $-a$ WLOG because we will restrict q in $f(x)$ to being positive WLOG (allowing p to be negative), which makes f symmetric about $x = 0$. Then a is bounded by $\frac{p}{q} < a < \frac{p+1}{q}$, where q is some positive integer and $p = \lfloor qa \rfloor / q$. We will choose some $q > 0$ such that the interval width $\frac{p+1}{q} - \frac{p}{q} = \frac{1}{q}$ is less than ϵ. Since q is merely some positive integer, this is totally possible; for instance, we could define $q = \lceil \frac{1}{\epsilon} \rceil$. Now consider the interval given by

$$I = \bigcap_{i=1}^{q} \left(\frac{\lfloor i \cdot a \rfloor}{i}, \frac{\lfloor i \cdot a \rfloor + 1}{i} \right)$$

All values in this interval are either irrational or rational, in which case they can be expressed in reduced form as r/k for $r, k \in \mathbb{Z}$. Note that $k \geq q$, because we've defined the interval as such to be in-between consecutive multiples of all reciprocal integers up to q, which requires a denominator larger than q to express. Now we will advance our argument. Let $\delta = \min(a - I_a, I_b - a)$ where I_a, I_b are the start and end of the interval I, respectively. Then all x such that $|x - a| < \delta$ falls in the interval I. We know that $f(a) = 0$ because a is irrational. If x is irrational, then $|f(x) - f(a)| = |0 - 0| = 0 < \epsilon$. If x is rational, then $|f(x) - f(a)| = |f(r/k)| = 1/k$. We know $1/k < \epsilon$ because $k \geq q$ and $\frac{1}{q} < \epsilon$ (as defined before). Therefore, we have shown that $f(x)$ is continuous for irrational x.

Now, we will show that $f(x)$ is discontinuous for all rational values of x. Let a be some nonzero rational number expressed in reduced form as r/k. By the same reasoning as above, if a is negative, we redefine a as $-a$ WLOG. Let δ be an arbitrary real. We will now consider x such that $|x - a| < \delta$. Let x be $a + \gamma$, where $\gamma = \delta/2$ if δ is irrational and $\delta/\sqrt{2}$ if δ is rational. This means that γ will always be irrational. Since a rational plus an irrational will always be irrational, this means that $a + \gamma$ will be irrational. Moreover, $a + \gamma$ satisfies the inequality $|x - a| < \delta$. $f(a + \gamma) = 0$ and $f(a) = \frac{1}{k}$, where $k > 0$. Then the inequality $|f(x) - f(a)| < \epsilon$ fails for $\epsilon = \frac{1}{2k}$. Therefore, $f(x)$ fails to be continuous for all nonzero rational values of x. Now, we will also show that $f(x)$ is discontinuous at $x = 0$. 0 is expressible as $0/k$, for any positive integer k. We follow the same reasoning as above and arrive at the inequality $|f(x) - f(a)| = |0 - k|$. Suppose $\epsilon = k/2$. Then the inequality does not hold. Therefore, $f(x)$ also fails to be continuous when $x = 0$.

Problem 6

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{y(y-x^2)}{x^4} & \text{if } 0 < y < x^2 \\ 0 & \text{otherwise} \end{cases}$$

Find the point(s) f is discontinuous.

We know from elementary calculus that $f(x) = x^2$ is continuous for all $\mathbb{R}$. From Theorems 1.10 and 1.11 in the textbook, we know that the operations of multiplication across two variables $f(x, y) = xy$ and subtraction across two variables $f(x, y) = x + y$ preserve continuity across $\mathbb{R}^2$, and that division across two variables $f(x, y) = x/y$ preserves continuity across $\mathbb{R}^2 \setminus \{(i, 0) : i \in \mathbb{R}\}$. This means that, for $0 < y < x^2$, $f(x, y)$ is continuous. $f(x, y)$ is not continuous at $x = 0$ because when $x = 0$, the denominator x^4 is 0.

There are two curves which separate the two pieces of the function: $y = 0$ and $y = x^2$. Note that as y approaches 0 from $y > 0$, $f(x, y)$ approaches zero, because it causes the numerator $y(y - x^2)$ to go to zero, which leads the entire expression to approach zero, even if x grows/shrinks while y approaches 0 (i.e. lateral movement). Moreover, $f(x, y)$ approaches zero as y approaches zero from $y < 0$ trivially, because the function is all zeroes. Therefore, f is continuous at all points along $y = 0$ (except $(0, 0)$). Note that as y approaches x^2 from $y < x^2$, the term $y - x^2$ goes to zero, which causes the entire function to go to zero. This is true even with lateral movement (i.e. x changes) because y still approaches the new value of x^2 during the descent and the term approaches zero. As before, $f(x, y)$ approaches zero as y approaches x^2 from $y > x^2$, trivially, because the function is all zeros. Therefore, the function f is continuous at all points along $y = x^2$ except for $(0, 0)$.

Hence, $f(x, y)$ is only discontinuous at all points where $x = 0$.