Homework 1

Andre Ye[1]

[1]University of Washington, MATH 334

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Problem 1

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ with $\vec{x} \neq 0 \neq \vec{y}$. The Cauchy-Schwarz inequality implies $\langle \vec{x}, \vec{y} \rangle \leq \|\vec{x}\| \cdot \|\vec{y}\|$. Suppose we have equality: $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \cdot \|\vec{y}\|$. We will show it follows that there exists some $\lambda \in \mathbb{R}$ such that $\vec{x} = \lambda \vec{y}$.

Squaring both sides of $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \cdot \|\vec{y}\|$ gives us $\langle \vec{x}, \vec{y} \rangle^2 = \|\vec{x}\|^2 \|\vec{y}\|^2$, which can be rewritten as $0 = \|\vec{x}\|^2 - \langle \vec{x}, \vec{y} \rangle^2 / \|\vec{y}|^2$. Recall that in the proof of the Cauchy-Schwarz inequality, the function $f(t) = \|\vec{x} - t\vec{y}\|^2$ is minimized at $t = \langle \vec{x}, \vec{y} \rangle / \|\vec{y}\|^2$, at which $f(t)$ can be written as $\|\vec{x}\|^2 - \langle \vec{x}, \vec{y} \rangle^2 / \|\vec{y}\|^2$. Given that $\|\vec{x}\|^2 - \langle \vec{x}, \vec{y} \rangle^2 / \|\vec{y}|^2$ is the minimum of$f(t)$, and that $\|\vec{x}\|^2 - \langle \vec{x}, \vec{y} \rangle^2 / \|\vec{y}|^2 = 0$, we conclude that $f(\langle \vec{x}, \vec{y} \rangle / \|\vec{y}\|^2) = \left\|\vec{x} - (\langle \vec{x}, \vec{y} \rangle / \|\vec{y}\|^2)\, \vec{y}\right\|^2 = 0$. If the norm of a vector is zero, then the vector itself must be the zero vector. Therefore, we know that $\vec{x} - (\langle \vec{x}, \vec{y} \rangle / \|\vec{y}\|^2)\, \vec{y} = 0$, or equivalently that $\vec{x} = (\langle \vec{x}, \vec{y} \rangle / \|\vec{y}\|^2) \vec{y}$. Therefore, there exists some $\lambda \in \mathbb{R}$ such that $\vec{x} = \lambda \vec{y}$, this λ being $\langle \vec{x}, \vec{y} \rangle / \|\vec{y}\|^2$. $\square$

Problem 2

Proof 1

Let $x, y \in \mathbb{R}^n$. Then, we will show that $2(\|x\|^2 + \|y\|^2) = \|x + y\|^2 + \|x - y\|^2$.

$$
\begin{aligned}
2(\|x\|^2 + \|y\|^2) &= 2(x \cdot x + y \cdot y) && \text{using } \|a\|^2 = a \cdot a \\
&= (x \cdot x + 2(x \cdot y) + y \cdot y) + (x \cdot x - 2(x \cdot y) + y \cdot y) && \text{adding cancellable terms} \\
&= \|x + y\|^2 + \|x - y\|^2. \quad \square && \text{grouping / ``un-distributing''}
\end{aligned}
$$

Proof 2

Let $x, y \in \mathbb{R}^n$. Then, we will show that $x \cdot y = \frac{\|x+y\|^2 - \|x-y\|^2}{4}$.

$$\begin{aligned}
x \cdot y &= \frac{1}{4} \cdot 4(x \cdot y) \\
&= \frac{1}{4}\left(2x \cdot y + 2x \cdot y\right) && \text{separation} \\
&= \frac{1}{4}\left((x \cdot x + 2x \cdot y + y \cdot y) - (x \cdot x - 2x \cdot y + y \cdot y)\right) && \text{adding cancellable terms} \\
&= \frac{1}{4}\left((x+y) \cdot (x+y) - (x-y) \cdot (x-y)\right) && \text{grouping} \\
&= \frac{\|x+y\|^2 - \|x-y\|^2}{4}. \quad \square && \text{using } a \cdot a = \|a\|^2
\end{aligned}$$

Problem 3

Suppose there are m vectors $x_1, ..., x_m \in \mathbb{R}^n$ which satisfy $\langle x_i, x_j \rangle = 0$ for all $i \neq j$. We will show it follows that $\| \sum_{i=1}^m x_i \|^2 = \sum_{i=1}^m \|x_i\|^2$ with a proof by induction.

Our inductive proposition $P(k)$ is that $\| \sum_{i=1}^k x_i \|^2 = \sum_{i=1}^k \|x_i\|^2$. We begin by showing the base case $P(1)$ holds. When $k = 1$, $P(k)$ reduces to $\|x_1\|^2 = \|x_1\|^2$, which is evidently true. Next, we will show that supposing $P(k)$ is true, $P(k+1)$ is necessarily true.

$$\begin{aligned}
P(k) &:= \|x_1 + ... + x_k\|^2 = \|x_1\|^2 + ... + \|x_k\|^2 \\
P(k+1) &:= \|x_1 + ... + x_k + x_{k+1}\|^2 = \|x_1\|^2 + ... + \|x_k|^2 + \|x_{k+1}\|^2
\end{aligned}$$

Let us group the terms in $P(k+1)$ to separate x_{k+1} from $x_1, ..., x_k$.

$$P(k+1) \iff \|(x_1 + ... + x_k) + x_{k+1}\|^2 = \left(\|x_1\|^2 + ... + \|x_k|^2\right) + \|x_{k+1}\|^2$$

Using $P(k)$, we can rewrite the first term of the LHS.

$$P(k+1) \iff \|(x_1 + ... + x_k) + x_{k+1}\|^2 = \left(\|x_1 + ... + x_k\|^2\right) + \|x_{k+1}\|^2$$

The norms can be rewritten as dot products, given that $\|a\|^2 = \langle a, a \rangle$.

$$P(k+1) \iff \langle (x_1 + ... + x_k) + x_{k+1}, (x_1 + ... + x_k) + x_{k+1} \rangle = \langle x_1 + ... + x_k, x_1 + ... + x_k \rangle + \|x_{k+1}\|^2$$

We will distribute the dot product in the LHS:

$$\begin{aligned}
&\langle (x_1 + ... + x_k) + x_{k+1}, (x_1 + ... + x_k) + x_{k+1} \rangle \\
=&\langle x_1 + ... + x_k, x_1 + ... + x_k \rangle + 2\langle x_1 + ... + x_k, x_{k+1} \rangle + \langle x_{k+1}, x_{k+1} \rangle
\end{aligned}$$

At this point, the LHS and RHS of the equivalence statement share two terms, $\langle x_1 + ... + x_k, x_1 + ... + x_k \rangle$ and $\langle x_{k+1}, x_{k+1} \rangle$ (equivalently written as $\|x_{k+1}\|^2$). Subtracting these terms from both sides (and dividing both sides by 2) yields the following equation:

$$\begin{aligned} P(k+1) &\iff \langle x_1 + ... + x_k, x_{k+1} \rangle = 0 \\ &\iff \langle x_1, x_{k+1} \rangle + ... + \langle x_k, x_{k+1} \rangle = 0 \end{aligned}$$

From the premise, $\langle x_i, x_j \rangle = 0$ for all $i \neq j$. Therefore, every term on the LHS is zero. The resulting expression, $0 = 0$, is true. Therefore, $P(k+1)$ is true, provided that $P(k)$ is true. Since $P(k)$ implies $P(k+1)$, and $P(k)$ holds for all $k \in \mathbb{N}$. $\square$

Problem 4

Given two real-valued functions on the unit interval $f, g : [0,1] \to \mathbb{R}$, the inner product is defined as $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. We will go through the proof of the Cauchy-Schwartz inequality and show that every step still works using this notion of the inner product between two real-valued functions, ultimately showing that $\left| \int_0^1 f(x)g(x)dx \right| \leq \left(\int_0^1 f(x)^2 dx \right)^{1/2} \left(\int_0^1 g(x)^2 dx \right)^{1/2}$.

To begin with, an inner product must be symmetric and bilinear. Firstly, $\int_0^1 f(x)g(x)dx = \int_0^1 g(x)f(x)dx$, which shows symmetry holds. Moreover, linearity is preserved: $\langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle$. In integrals, this is equivalent to stating that $\int_0^1 f(x)(g(x)+h(x))dx = \int_0^1 f(x)g(x)dx + \int_0^1 f(x)h(x)dx$. This is true because the integral itself is linear. Bilinearity follows trivially from symmetry and linearity.

If $g(x) = 0$, both sides of the inequality are zero. Otherwise, we introduce a real variable t and consider a function across functions q quadratic in t, $q(t) = |f - tg|^2 = \langle f - tg, f - tg \rangle = \langle f, f \rangle - 2t\langle f, g \rangle + t^2 \langle g, g \rangle$. The minimum using the quadratic formula can be found at $t = \frac{\langle f, g \rangle}{\langle g, g \rangle}$, which yields the value $q(t) = \langle f, f \rangle - \frac{\langle f, g \rangle^2}{\langle g, g \rangle}$. Because the absolute value is zero or positive, and squaring a zero or positive value will always yield a value of zero or positive value, $q(t) = |f - tg|^2 \geq 0$. This means that the value at the minimizing value of t will also be greater than or equal to zero: $\langle f, f \rangle - \frac{\langle f, g \rangle^2}{\langle g, g \rangle} \geq 0$. Multiplying both sides by $\langle g, g \rangle$ and rearranging yields $\langle f, f \rangle \cdot \langle g, g \rangle \geq \langle f, g \rangle^2$. Since $\langle a, a \rangle = \|a\|^2$, we can rewrite this as $\|f\|^2 \|g\|^2 \geq \langle f, g \rangle^2$. Taking the square root of both sides yields $\langle f, g \rangle \leq \|f\| \|g\|$. Rewriting this in terms of the norm definition of the inner product yields $\left| \int_0^1 f(x)g(x)dx \right| \leq \left(\int_0^1 f(x)^2 dx \right)^{1/2} \left(\int_0^1 g(x)^2 dx \right)^{1/2}$. Therefore, all of the steps of Cauchy-Schwartz hold under the inner product across functions. $\square$

Problem 5

Problem 5a.

We will show that, for all $n \in \mathbb{N}$, $1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2$ by induction.

Our inductive proposition $P(k)$ is that $1^3 + 2^3 + ... + k^3 = (1 + 2 + ... + k)^2$. The base case $P(1)$ holds: $1^3 = 1^2 \implies 1 = 1$. Next, we will show that if $P(k)$ is true, $P(k+1)$ is necessarily true.

$$\begin{aligned} P(k) &= 1^3 + 2^3 + ... + k^3 = (1 + 2 + ... + k)^2 \\ P(k+1) &= 1^3 + 2^3 + ... + k^3 + (k+1)^3 = (1 + 2 + ... + k + (k+1))^2 \end{aligned}$$

Expanding the RHS of $P(k+1)$ yields

$$(1^3 + 2^3 + ... + k^3) + (k+1)^3 = (1 + 2 + ... + k)^2 + 2(1 + 2 + ... + k)(k+1) + (k+1)^2$$

By $P(k)$, the first term of the RHS and the first term of the RHS are equivalent, and both can be subtracted from the equation.

$$(k+1)^3 = 2(1 + 2 + ... + k)(k+1) + (k+1)^2$$

We will endeavor in laborious algebra to prove the equality:

$$\begin{aligned} (k+1)^3 &= 2(1 + 2 + ... + k)(k+1) + (k+1)^2 \\ &= 2\left(\frac{k(k+1)}{2}\right)(k+1) + k^2 + 2k + 1 \\ &= k(k+1)^2 + k^2 + k + 1 \\ &= k(k^2 + 2k + 1) + k^2 + 2k + 1 \\ &= k^3 + 2k^2 + k + k^2 + 2k + 1 \\ &= k^3 + 3k^2 + 3k + 1 \\ &= (k+1)^3 \end{aligned}$$

Therefore, $P(k)$ implies $P(k+1)$, and $P(k)$ holds for all $k \in \mathbb{N}$. $\square$

Problem 5b.

We will show that a set $\{a_1, a_2, ..., a_n\}$ has 2^n subsets, of which 2^{n-1} have an even number of elements and 2^{n-1} have an odd number of elements, using induction.

Our inductive proposition $P(n)$ is that an arbitrary set of cardinality n has 2^n subsets of which 2^{n-1} have an even cardinality and the other 2^{n-1} have an odd cardinality. We will show $P(n)$ holds across $n \in \mathbb{N}$. In the base case $P(1)$, for a set of size 1 $\{x_1\}$, there are 2^1 subsets, the empty set $\emptyset$ (even cardinality) and $\{x_1\}$ (odd cardinality). There are 2^{1-1} even cardinality subsets and 2^{1-1} odd cardinality subsets. Therefore, $P(1)$ holds. Next, we will show that $P(k)$ implies $P(k+1)$. Suppose we have a set of cardinality $k+1$. This is equivalent to a set of cardinality k, $\{x_1, x_2, ..., x_k\}$ with an additional element x_{k+1}. By the inductive proposition, the set of cardinality k has 2^{k-1} subsets with an even cardinality and 2^{k-1} subsets with an odd cardinality. The set of all subsets of a cardinality-$k+1$ set can be partitioned by the set of odd-cardinality subsets and the set of even-cardinality subsets. In turn, each of these sets can be partitioned by the set of odd/even-cardinality subsets including x_{k+1} and the set of odd/even-cardinality subsets not including x_{k+1}. For each set with an even cardinality, we can add x_{k+1}. This generates 2^{k-1} unique sets with an odd cardinality, since adding one element to a set with even cardinality makes it odd. Likewise, for each set with an odd cardinality, we can add x_{k+1}. This generates 2^{k-1} unique sets with an even cardinality. In total, this yields $2^{k-1} + 2^{k-1} = 2^k$ odd-cardinality subsets and $2^{k-1} + 2^{k-1} = 2^k$ even-cardinality subsets. We have shown that $P(k)$ implies $P(k+1)$. Therefore, by induction, $P(n)$ holds across $n \in \mathbb{Z}^+$. $\square$

Problem 6

Let A be a finite set with an odd number of elements n. Assume furthermore that $f : A \to A$ is a function satisfying $f(f(a)) = a$ for all $a \in A$a. We will show that there exists at least one element satisfying $f(a) = a$.

Suppose A has only 1 element. Then it trivially follows that $f(a) = a$, because there is only one possible term to map from and map to. Suppose A has more than 1 elements. Let x_1 be some element in A. Let $f(x_1) = x_2$, for some $x_2 \in A$. Moreover, recall that $f(f(x_1)) = x_1$ must be true, per the definition of f. Then, we have that $f(x_2) = x_1$. Let x_3 be an element in A distinct from both x_1 and x_2. $f(x_3)$ cannot have the same value as x_2 – if this were the case, then $f(f(x_3)) = x_1 \neq x_3$; likewise, $f(x_3)$ cannot be equal to x_1, as if this were the case, then $f(f(x_1)) = x_2 \neq x_3$. Therefore, $f(x_3)$ must map to an element of A distinct from either x_1 or x_2. Call this element x_4. Using the same reasoning as above, invoking the property that $f(f(a)) = a$, we have that $f(x_3) = x_4$ and $f(x_4) = x_3$. In general, consider adding distinct elements in pairs x_{2k}, x_{2k+1} for $k \in \mathbb{Z}^+ \cup \{0\}$. By the same reasoning above, both elements must be distinct from all elements in $\{x_i : i \in \mathbb{Z}^+, i < k\}$ to preserve the properties that $f(f(a)) = a$ for all elements in this set. Moreover, $f(x_{2k}) = x_{2k+1}$ and $f(x_{k+1}) = x_{2k}$, using the same reasoning above.. In the end, we only have one more element x_n left in A. We will always have one element left because A is odd, whereas we have hitherto only covered the elements of A in pairs. Using the same reasoning as above,

$f(x_n)$ cannot map to any other element in A, as then the criteria that $f(f(x_n)) = x_n$ would not be true. Therefore, since $f(x_n)$ must map to some element in A, but cannot map to any element in $A \setminus \{x_n\}$, $f(x_n)$ must be equal to x_n. $\square$

In the context of a ballroom dance, this means that in an odd group of people who are told to dance in pairs, such that if x dances with y then y must also dance with x, then one individual will be left utterly alone, without anyone else to dance with, except themselves.

Problem 7

Let $f(x) = x$ if x is rational and $f(x) = 0$ if x is irrational. We will show that f is continuous at $x = 0$ and nowhere else.

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point a if $\lim_{x \to a} f(x) = f(a)$. This means that if a function were continuous, then $\forall \epsilon_{\in \mathbb{R}^+} : \exists \delta_{\in \mathbb{R}^+}$ such that $|f(x) - f(a)| < \epsilon$, for $0 < |x - a| < \delta$. Let us consider an arbitrary $\epsilon \in \mathbb{R}^+$. To show that f is continuous at $a = 0$, we would need to demonstrate that there exists some $\delta \in \mathbb{R}^+$ such that $|f(x) - f(0)| = |f(x)| < \epsilon$ for $0 < |x| < \delta$. Let $\delta = \epsilon/2$. Then x is constrained to the interval $(0, \epsilon/2)$. $f(x)$ will either be 0 if x is irrational and x if x is rational. If x is irrational, then $|f(x)| < \epsilon$ holds, as $0 < \epsilon$. If x is rational, then $|f(x)| < \epsilon$ still holds, as $\epsilon/2 < \epsilon$, $\epsilon/2$ being an upper bound on the possible value of x. Therefore, for all ϵ, there exists some δ (defined in terms of ϵ) such that for x where $0 < |x| < \delta$, $|f(x)| < \epsilon$ holds. This demonstrates that f is continuous at $x = 0$.

On the other hand, we need to show that f is discontinuous at nonzero a. That is, if a is not zero, f is not continuous. Consider an arbitrary $\epsilon \in \mathbb{R}^+$ and $a \in \mathbb{R} \setminus \{0\}$. Suppose a is rational. Then, if f is continuous at a, $\forall \epsilon_{\in \mathbb{R}^+} : \exists \delta_{\in \mathbb{R}^+}$ s.t. $|f(x) - a| < \epsilon$, for $0 < |x - a| < \delta$. Suppose $\epsilon = |a/2|$. Let δ be an arbitrary real positive number, and γ be a positive irrational less than δ. Then setting $x = a + \gamma$ satisfies $|(a + \gamma) - a| < \delta$. Because a rational added to an irrational is an irrational, $f(a + \gamma) = 0$. Then, it does not hold that $\forall \epsilon_{\in \mathbb{R}^+} |0 - a| < \epsilon$ for nonzero a, because there is no $\delta \in \mathbb{R}^+$ such that the statement $|0 - a| < \epsilon$ holds for $\epsilon = |a/2|$, which is $|a| < |a/2|$. Therefore, if a is rational and nonzero, f is not continuous at a.

Now, suppose a is irrational. If f is continuous at a, $\forall \epsilon_{\in \mathbb{R}^+} : \exists \delta_{\in \mathbb{R}^+}$ s.t. $|f(x)| < \epsilon$, for $0 < |x - a| < \delta$. Suppose again that $\epsilon = |a/2|$, and δ be any positive real number. Let x be a nonzero rational number chosen from the interval $(a, a + \delta)$ if x is positive or $(a - \delta, a)$ if x is negative. Then the interval $0 < |x - a| < \delta$ is necessarily satisfied, yet since $f(x) = x$ as x is rational and x is nonzero, $|f(x)| = |x| < \epsilon$ does not hold. Since $|a| = 2\epsilon$ and $x = a + \gamma$ where $0 < \gamma < \delta$, $|f(x)| = |x| < \epsilon$ reduces to $2\epsilon + \gamma < \epsilon$, which is clearly not true. As such, there is no value of δ such that for $0 < |x - a| < \delta$, $|f(x)| < \epsilon$ holds for

$\epsilon = |a/2|$. Therefore, if a is irrational and nonzero, f is not continuous at a. Hence, f is continuous at $x = 0$ and discontinuous everywhere else. $\square$