Conceptual Problem Set 2, MATH 208

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Ch. 1, Problem 7

a. Use Gauss-Jordan elimination to find the general solution for the following system of linear equations.

We can begin by expressing the linear system as an augmented matrix:

$$\left(\begin{array}{cccc|c} 0 & 1 & 3 & -1 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ -2 & -4 & 4 & -2 & 0 \end{array}\right)$$

Let us execute a sequence of operations to clean up the matrix for convenience: $R_1 \iff R_3$, $R_2 \iff R_3$, $R_1 \cdot -1/2 \to R_1, R_2 \cdot -1 \to R_2$.

$$\left(\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -1 & 0 \end{array}\right)$$

We can eliminate the first column with $R_1 - R_2 \to R_2$:

$$\left(\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & 3 & -1 & 0 \end{array}\right)$$

The second column can be cleared with $R_2 - R_3 \to R - 3$ followed by $R_1 - 2 \cdot R_2 \to R_1$:

$$\left(\begin{array}{cccc|c} 1 & 0 & 4 & -3 & 0 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & -6 & 3 & 0 \end{array}\right)$$

Let us scale the third row by $-1/6$:

$$\left(\begin{array}{cccc|c} 1 & 0 & 4 & -3 & 0 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1/2 & 0 \end{array}\right)$$

We will clear the third column with $R_2 + 3 \cdot R_3 \to R_2$ and $R_1 - 4 \cdot R_3 \to R_1$.

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1/2 & 0 \end{array}\right)$$

Let $x_4 = s_1$; then we have the following solution:

$$x_1 = s_1$$
$$x_2 = -{}^1/_2 s_1$$
$$x_3 = {}^1/_2 s_1$$
$$x_4 = s_1$$

b. Give an example of a nonzero solution to the previous system of linear equations.

We can use $s_1 = \pi$ to derive one possible nonzero solution:

$$(\pi, -{}^\pi/_2, {}^\pi/_2, \pi)$$

c. The points $(1, 0, 3), (1, 1, 1)$, *and* $(-2, -1, 2)$ *lie on a unique plane* $a_1x_1 + a_2x_2 + a_3x_3 = b$. *Using your previous answers, find an equation for this plane.*

Given these points and the general equation for a plane, we have the following linear system:

$$1a_1 + 0a_2 + 3a_3 = b$$
$$1a_1 + 1a_2 + 1a_3 = b$$
$$-2a_1 - 1a_2 + 2a_3 = b$$

This system can be expressed as the following augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & -1 & 0 \end{array}\right)$$

Let us rearrange the columns with $C_1 \iff C_2$; this is valid as long as we remember the column correspondence: $\{C_1 : z_2, C_2 : z_1, C_3 : z_3, C_4 : b\}$.

$$\left(\begin{array}{cccc|c} 0 & 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ -1 & -2 & 2 & -1 & 0 \end{array}\right)$$

We will perform two more scaling operations: $C_2 \cdot -1 \to C_2$ and $C_3 \cdot 2 \to C_3$.

$$\left(\begin{array}{cccc|c} 0 & 1 & 3 & -1 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ -2 & -4 & 4 & -2 & 0 \end{array}\right)$$

This is the same matrix as the augmented matrix derived in part a) of this problem. We know that the general system is given by $(s_1, -{}^{s_1}/_2, {}^{s_1}/_2, s_1)$; therefore we can express the solution to the plane construction system presented in this problem as $z_1 = -{}^{s_1}/_2, z_2 = s_1, z_3 = {}^{s_1}/_2, b = s_1$. Let $s_1 = 1$; then one possible equation for a plane containing the three points will be

$$\boxed{-\frac{1}{2}x_1 + x_2 + \frac{1}{2}x_3 = 1}$$

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Ch. 1, Problem 8

a. Find the echelon form of the augmented matrix of the above system.
The augmented matrix for this system is

$$\left(\begin{array}{ccc|c} 2 & 3 & -5 & b_1 \\ 7 & 2 & 8 & b_2 \\ -1 & 1 & -5 & b_3 \end{array}\right)$$

We will clear the first column with $2R_2 - 7R_1 \to R_2$ and $R_1 + 2R_3 \to R_3$.

$$\left(\begin{array}{ccc|c} 2 & 3 & -5 & b_1 \\ 0 & -17 & 51 & b_2 \\ 0 & 5 & -15 & b_3 \end{array}\right)$$

The second column can be cleared with $5R_2 + 17R_3 \to R_3$, which will yield a matrix in echelon form.

$$\boxed{\left(\begin{array}{ccc|c} 2 & 3 & -5 & b_1 \\ 0 & -17 & 51 & b_2 \\ 0 & 0 & 0 & b_3 \end{array}\right)}$$

b. Find the conditions on (b_1, b_2, b_3) for which this system has a solution.

From R_3 of the echelon-form matrix, we see that $b_3 = 0$ is necessary for the system to be consistent. Given this, the equations represented by R_1 and R_2 simplify to:

$$2x_1 + 3x_2 = b_1$$
$$-17x_2 = b_2$$

This system is consistent for any real-valued b_1 and b_2. We may prove it as follows: for any $\{b_1, b_2\} \in \mathbb{R}^2$, the solution is given through simple linear algebraic rearrangement by $x_2 = -\frac{b_2}{17}$ and $x_1 = \frac{b_1 + 3 \cdot \frac{b_2}{17}}{2}$, which are defined for all possible inputs. Done.
Therefore, the necessary constraints are as follows:

$$\boxed{\begin{array}{c} b_1 \in \mathbb{R} \\ b_2 \in \mathbb{R} \\ b_3 = 0 \end{array}}$$

c. Do you see the shape of the points (b_1, b_2, b_3) for which the above system has a solution?

There are two free variables; therefore it forms a $\boxed{\text{plane}}$.

d. If you randomly picked a (b_1, b_2, b_3) in $\mathbb{R}^3$, do you expect the above system to have a solution?

$\boxed{\text{No}}$, because the solution space is a a two-dimensional plane whereas the sampling space is three-dimensional.

Ch. 1, Problem 9

a. For what values of a and b does the system have infinitely many solutions?

The augmented matrix is given by:

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 2 & 0 & -8 & -2 \\ -6 & 6 & a & b \end{array}\right)$$

Let us begin by putting the matrix in echelon form. We will clear the first column with $R_1 \cdot 2 - R_2 \rightarrow R_2$ and $R_1 \cdot 6 + R_3 \rightarrow R_3$:

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -6 & -6 & 6 \\ 0 & -12 & 6+a & 24+b \end{array}\right)$$

We will the second row by $-\frac{1}{6}$ for convenience.

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & -12 & 6+a & 24+b \end{array}\right)$$

Next, we will clear the second column with $R_2 \cdot 12 + R_3 \cdot R_3$ and $R_1 + 3 \cdot R_2 \rightarrow R_1$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 18+a & 12+b \end{array}\right)$$

For now, we will pause manipulation. A system represented as a matrix has an infinite number of solutions when there is at least one zero-row; we can create this with

$$\boxed{a = -18; b = -12}$$

b. Give an example of a and b where the system has exactly one solution.

Recall that the modified matrix is as such:

$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 18+a & 12+b \end{array}\right)$$

This is a system of three equations with three variables; thus almost any combination of a and b will yield a solution (assuming $\neg((a = -18) \wedge (b = -12)$, among other conditions which would lead to case c). For instance, let $\boxed{a = -17 \text{ and } b = -11}$ for convenience.

One can easily derive a solution from this system.

$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

c. Give an example of a and b for which the system has no solutions.

Recall that the modified matrix is as such:

$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 18+a & 12+b \end{array}\right)$$

Let $a = -18$ and b be an arbitrary real which is not equal to -12, e.g. π. Then the third row's equation representation becomes $0x_3 = 12 + \pi$, for which no real value of x_3 satisfies.

Ch. 2, Problem 1

a. Last week, Jake spent 10 hours working from home, 15 hours working in his office in Padelford Hall, and 2 hours working at Cafe Allegro. Compute what was accomplished, and express the result as a vector equation.

Let y_r represent the amount of time spent on research, let y_t be the time spent on teaching, and let y_c be the cups of coffee consumed. Moreover, let x_h represent the amount of time working from home, x_o be the time working from the office, and x_a be the time working from Cafe Allegro. All units of time are in hours. We have the following equations:

$$\frac{2}{3}x_h + \frac{1}{3}x_o + \frac{5}{12}x_a = y_r$$
$$\frac{1}{6}x_h + \frac{1}{2}x_o + \frac{5}{12}x_a = y_t$$
$$\frac{1}{2}x_h + 0x_o + 1x_a = y_c$$

This can be expressed as the following vector equation:

$$\begin{pmatrix} 2/3 \\ 1/6 \\ 1/2 \end{pmatrix} x_h + \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix} x_0 + \begin{pmatrix} 5/12 \\ 5/12 \\ 1 \end{pmatrix} x_a = \begin{pmatrix} y_r \\ y_t \\ y_c \end{pmatrix}$$

In this case, we have $x_h = 10$, $x_0 = 15$, and $x_a = 2$:

$$10 \begin{pmatrix} 2/3 \\ 1/6 \\ 1/2 \end{pmatrix} + 1 \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 5/12 \\ 5/12 \\ 1 \end{pmatrix} = \begin{pmatrix} y_r \\ y_t \\ y_c \end{pmatrix}$$

$$\begin{pmatrix} 20/3 \\ 5/3 \\ 5 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 5/6 \\ 5/6 \\ 2 \end{pmatrix} = \begin{pmatrix} y_r \\ y_t \\ y_c \end{pmatrix}$$

$$\begin{pmatrix} 47/6 \\ 18/6 \\ 7 \end{pmatrix} = \begin{pmatrix} y_r \\ y_t \\ y_c \end{pmatrix}$$

Therefore, Jake got done

$47/6$ hrs. of research
$\frac{9}{3}$ hrs. of teaching
7 cups of coffee

b. This week, Jake has 15 hours of research to work on and 10 hours of work related to teaching. He also wants 11 cups of coffee, because... of... very important reasons. How much time should he spend working from home, from his office, and from the coffeeshop? .

Putting aside the dubious health effects of drinking eleven cups of coffee, we can use the following equation:

$$\begin{pmatrix} 2/3 \\ 1/6 \\ 1/2 \end{pmatrix} x_h + \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix} x_0 + \begin{pmatrix} 5/12 \\ 5/12 \\ 1 \end{pmatrix} x_a = \begin{pmatrix} 15 \\ 10 \\ 11 \end{pmatrix}$$

This can be represented as the following augmented matrix:

$$\left(\begin{array}{ccc|c} 2/3 & 1/3 & 5/12 & 15 \\ 1/6 & 1/2 & 5/12 & 10 \\ 1/2 & 0 & 1 & 11 \end{array}\right)$$

Let us perform some scaling for the convenience of calculations.

$$\left(\begin{array}{ccc|c} 8 & 4 & 5 & 180 \\ 2 & 6 & 5 & 120 \\ 1 & 0 & 2 & 22 \end{array}\right)$$

We will eliminate the first column using $R_1 - 4 \cdot R_2 \to R_2$ and $R_1 - 8 \cdot R_3 \to R_3$.

$$\left(\begin{array}{ccc|c} 8 & 4 & 5 & 180 \\ 0 & -20 & -15 & -300 \\ 0 & 4 & -11 & 4 \end{array}\right)$$

The second column can be cleared with $R_2 + 5 \cdot R_3 \to R_3$:

$$\left(\begin{array}{ccc|c} 8 & 4 & 5 & 180 \\ 0 & -20 & -15 & -300 \\ 0 & 0 & -70 & -280 \end{array}\right)$$

From this augmented matrix, we understand that $x_a = 4$. Back-substituting into the second row: $-20x_o - 15 \cdot 4 = -300 \implies -20x_o - 60 = -300 \implies -20x_o = -240 \implies x_o = 12$. Finally, back-substituting into the third row: $8x_h + 4 \cdot 12 + 5 \cdot 4 = 180 \implies 8x_h + 48 + 20 = 180 \implies 8x_h + 68 = 180 \implies 8x_h = 112 \implies x_h = 14$. Thus, in order to accomplish the goals outlined in the problem, Jake must spend

14 hours from home
12 hours from the office
4 hours from Cafe Allegro

c. Describe the situation in (b) as a vector equation and a matrix equation $A\mathbf{t} = \mathbf{w}$. What do the vectors $\mathbf{t}$ and $\mathbf{w}$ mean in this context? For which other vectors $\mathbf{w}$ does the equation $A\mathbf{t} = \mathbf{w}$ have a solution?

We needed to solve the following vector equation:

$$\begin{pmatrix} 2/3 \\ 1/6 \\ 1/2 \end{pmatrix} x_h + \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix} x_o + \begin{pmatrix} 5/12 \\ 5/12 \\ 1 \end{pmatrix} x_a = \begin{pmatrix} 15 \\ 10 \\ 11 \end{pmatrix}$$

This can be expressed as the following matrix equation:

$$\begin{pmatrix} 2/3 & 1/3 & 5/12 \\ 1/6 & 1/2 & 5/12 \\ 1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_h \\ x_o \\ x_a \end{pmatrix} = \begin{pmatrix} 15 \\ 10 \\ 11 \end{pmatrix}$$

Here, $\mathbf{t} = \begin{pmatrix} x_h \\ x_o \\ x_a \end{pmatrix}$ represents the various times that can be spent at each location and $\mathbf{w} = \begin{pmatrix} 15 \\ 10 \\ 11 \end{pmatrix}$ represents the work that needs to be done.

In order to determine for which other vectors $\mathbf{w}$ the equation $A\mathbf{t} = \mathbf{w}$ has a solution, we need to determine the span of the following vector set:

$$\left\{ \begin{pmatrix} 2/3 \\ 1/6 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 5/12 \\ 5/12 \\ 1 \end{pmatrix} \right\}$$

We can do this by applying Gaussian elimination to the augmented matrix A and observing the solution field. We can start by scaling the matrix for convenience: $R_1 \cdot 12 \to R_1$, $R_2 \cdot 12 \to R_2$, and $R_3 \cdot 2 \to R_3$.

$$\left(\begin{array}{ccc|c} 8 & 4 & 5 & a \\ 2 & 6 & 5 & b \\ 1 & 0 & 2 & c \end{array}\right)$$

The first column can be cleared with $R_1 - 4 \cdot R_2 \to R_2$ and $R_1 - 8 \cdot R_3 \to R_3$:

$$\left(\begin{array}{ccc|c} 8 & 4 & 5 & a \\ 0 & -20 & -15 & a - 4b \\ 0 & 4 & -3 & a - 8c \end{array}\right)$$

The second column can be cleared with $R_2 + 5 \cdot R_3 \to R_3$:

$$\left(\begin{array}{ccc|c} 8 & 4 & 5 & a \\ 0 & -20 & -15 & a - 4b \\ 0 & 0 & -30 & 6a - 4b - 40c \end{array}\right)$$

Given any $\{a, b, c\} \in \mathbb{R}^3$, we can solve $A\mathbf{t} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Therefore, $\boxed{\text{any vector } \mathbf{w} \in \mathbb{R}^3 \text{ works}}$.

d. Jake tries working in the math department lounge for an hour, and gets 30 minutes of research and 20 minutes of teaching work done, while having time to drink $\frac{1}{3}$ of a cup of coffee. Not bad. But Jake's colleague Vasu claims that there's no need to work in the lounge – the other options already give enough flexibility. Is he right? Explain mathematically.

The amount of work Jake completes can be represented as $\mathbf{w} = \begin{pmatrix} 1/2 \\ 1/3 \\ 1/3 \end{pmatrix} \in \mathbb{R}^3$. From the result derived in part c., we know that there is some combination of time Jake can spend between home, the math department, and the coffeeshop such that he can accomplish the same amount of work. Mathematically, the vector representing the amount of work Jake got done in the lounge is within the span of the amount of work Jake could have gotten done at home, in the department, and at the cofeeshop.

Ch. 2, Problem 3

We are given three vectors in $\mathbb{R}^3$: $\langle 2, -1, 3\rangle, \langle 1, 2, 2\rangle, \langle -4, z_1, z_2\rangle$; we must find all values for z_1 and z_2 such that the span of the three vectors is not $\mathbb{R}^3$.

We can reduce the span to a two-dimensional plane if $\langle -4, z_1, z_2\rangle$ is a linear combination of $\langle 2, -1, 3\rangle$ and $\langle 1, 2, 2\rangle$. Let us parameterize the solution space with a variable a. We have the following:

$$\langle -4, z_1, z_2\rangle = a\langle 2, -1, 3\rangle + (-4 - 2a)\langle 1, 2, 2\rangle$$

From this, we can infer the following solutions for z_1 and z_2:

$$z_1 = -a - 8 - 4a = \boxed{-5a - 8}$$
$$z_2 = 3a - 8 - 4a = \boxed{-a - 8}$$

Ch. 2, Problem 4

a. Find two vectors $\mathbf{u}_1, \mathbf{u}_2 \in R^3$ so that span$\{\mathbf{u}_1, \mathbf{u}_2\} = P$, where P is a plane defined by $2x_1 - x_2 + 4x_3 = 0$.

Let $\mathbf{u}_{m,n}$ refer to the nth element of the vector $\mathbf{u}_m$. Let $a, b \in \mathbb{R}$. We have the following relationship between the vector values and the plane they span:

$$2(a\mathbf{u}_{1,1} + b\mathbf{u}_{2,1}) - (a\mathbf{u}_{1,2} + b\mathbf{u}_{2,2}) + 4(a\mathbf{u}_{1,3} + b\mathbf{u}_{2,3}) = 0$$

This can be expanded and simplified:

$$2a\mathbf{u}_{1,1} + 2b\mathbf{u}_{2,1}) - a\mathbf{u}_{1,2} - b\mathbf{u}_{2,2}) + 4a\mathbf{u}_{1,3} + 4b\mathbf{u}_{2,3} = 0$$
$$a(2\mathbf{u}_{1,1} - \mathbf{u}_{1,2} + 4\mathbf{u}_{1,3}) + b(2\mathbf{u}_{2,1} - \mathbf{u}_{2,2} + 4\mathbf{u}_{2,3}) = 0$$
$$a(2\mathbf{u}_{1,1} - \mathbf{u}_{1,2} + 4\mathbf{u}_{1,3}) = -b(2\mathbf{u}_{2,1} - \mathbf{u}_{2,2} + 4\mathbf{u}_{2,3})$$

We can satisfy this equation for any value of a and b if $2\mathbf{u}_{1,1} - \mathbf{u}_{1,2} + 4\mathbf{u}_{1,3} = 0$ and $2\mathbf{u}_{2,1} - \mathbf{u}_{2,2} + 4\mathbf{u}_{2,3} = 0$. Two possible vectors which satisfy this condition are

$$\mathbf{u}_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}$$

b. Consider three vectors $\mathbf{u}_1 = \langle 2, 7, -1\rangle, \mathbf{u}_2 = \langle 3, 2, 1\rangle, \mathbf{u}_3 = \langle -5, 8, -5\rangle$. Let $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ be an arbitrary vector in $\mathbb{R}^3$. Use Gaussian elimination to determine which vectors $\mathbf{b}$ are in span$\{\mathbf{u}_1, \mathbf{u_2}, \mathbf{u_3}\}$. Without further calculation, conclude that span$\{\mathbf{u}_1, \mathbf{u_2}, \mathbf{u_3}\}$ is a plane in $\mathbb{R}^3$ and identify an equation of the plane in the form $ax_1 + bx_2 + cx_3 = 0$.

We have the following augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 3 & -5 & b_1 \\ 7 & 2 & 8 & b_2 \\ -1 & 1 & -5 & b_3 \end{array}\right)$$

We can eliminate the first column with $7R_1 - 2R_2 \to R_2$ and $R_1 + 2R_3 \to R_3$.

$$\left(\begin{array}{ccc|c} 2 & 3 & -5 & b_1 \\ 0 & 17 & -51 & 7b_1 - 2b_2 \\ 0 & 5 & -15 & b_1 + 2b_3 \end{array}\right)$$

We will scale with $R_2 \cdot \frac{1}{17} \to R_2$ and $R_3 \cdot \frac{1}{5} \to R_3$ for convenience:

$$\left(\begin{array}{ccc|c} 2 & 3 & -5 & b_1 \\ 0 & 1 & -3 & \frac{7b_1-2b_2}{17} \\ 0 & 1 & -3 & \frac{b_1+2b_3}{5} \end{array}\right)$$

Now, we will eliminate the second column with $R_2 - R_3 \to R_3$:

$$\left(\begin{array}{ccc|c} 2 & 3 & -5 & b_1 \\ 0 & 1 & -3 & \frac{7b_1-2b_2}{17} \\ 0 & 0 & 0 & \frac{7b_1-2b_2}{17} - \frac{b_1+2b_3}{5} \end{array}\right)$$

This matrix is in echelon form. A vector is in the span if the following is satisfied:

$$\frac{7b_1 - 2b_2}{17} - \frac{b_1 + 2b_3}{5} = 0$$

The span is a two-dimensional plane in $\mathbb{R}^3$: we can observe that the constraint expressed above can be expressed as $b_n = ...$ for any $n \in \{1, 2, 3\}$, since the relationship is linear; in this formulation, two variables are fixed and the third (b_n) is derived as a linear combination of these two. In fact, simplifying this expression yields the following plane:

$$\frac{18}{85}b_1 - \frac{2}{17}b_2 - \frac{2}{5}b_3 = 0$$