Conceptual Problem Set 1, MATH 208

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Problem 1

a. *Write an augmented matrix for the system of equations describing the length of each task.*

We have the following equations:

$$2t_1 + 2t_2 + 2t_3 = t_4 + t_5 \implies 2t_1 + 2t_2 + 2t_3 - t_4 - t_5 = 0$$
$$2t_2 + 2t_3 + 2t_4 = t_5 + t_6 \implies 2t_2 + 2t_3 + 2t_4 - t_5 - t_6 = 0$$
$$t_2 = 1$$
$$t_4 = 10$$

We can form the following augmented matrix:

$$\left(\begin{array}{cccccc|c} 2 & 2 & 2 & -1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 2 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \end{array}\right)$$

b. *Reduce this augmented matrix to reduced echelon form.*

We begin with $R_2 - 2 \cdot R_3 \to R_3$:

$$\left(\begin{array}{cccccc|c} 2 & 2 & 2 & -1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 2 & -1 & -1 & 0 \\ 0 & 0 & 2 & 2 & -1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \end{array}\right)$$

Next, we will clear all values in the fourth column above the fourth row with $R_3 - 2 \cdot R_4 \to R_3$, $R_2 - 2 \cdot R_4 \cdot R_2$, and $R_1 + R_2 \to R_1$:

$$\left(\begin{array}{cccccc|c} 2 & 2 & 2 & 0 & -1 & 0 & 10 \\ 0 & 2 & 2 & 0 & -1 & -1 & -20 \\ 0 & 0 & 2 & 0 & -1 & -1 & -22 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \end{array}\right)$$

Clearing all values in the third column above the third row with $R_2 - R_3 \to R_2$ and $R_1 - R_3 \to R_1$:

$$\left(\begin{array}{cccccc|c} 2 & 2 & 0 & 0 & 0 & 1 & 32 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & -1 & -1 & -22 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \end{array}\right)$$

Lastly, clearing the value at position $(1, 2)$ with $R_1 - R_2 \to R_1$:

$$\left(\begin{array}{cccccc|c} 2 & 0 & 0 & 0 & 0 & 1 & 30 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & -1 & -1 & -22 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \end{array}\right)$$

The last step is to multiply R_1, R_2, and R_3 by the reciprocal of the leading term such that all pivot values are 1:

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 15 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -11 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \end{array}\right)$$

This is the final reduced echelon form.

c. Suppose he knows additionally that the sixth task will take 20 seconds and the first three tasks together will take 50 seconds. Write the extra rows that you would add to your answer in (b) to take account of this new information.

The following additional information is provided:

$$t_6 = 20$$

$$t_1 + t_2 + t_3 = 50$$

We can form the following augmented matrix:

$$\left(\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 0 & 1 & 20 \\ 1 & 1 & 1 & 0 & 0 & 0 & 50 \end{array}\right)$$

We can append this to the original augmented matrix as follows:

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 50 \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 15 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -11 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 20 \end{array}\right)$$

To put into echelon form, we will use

$$\left(\begin{array}{cccccc|c} 0 & 1 & 1 & 0 & 0 & -\frac{1}{2} & 35 \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 15 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -11 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 20 \end{array}\right) \to \left(\begin{array}{cccccc|c} 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & 34 \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 15 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -11 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 20 \end{array}\right)$$

$$\to \left(\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 45 \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 15 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -11 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 20 \end{array}\right) \to \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 15 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -11 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 0 & 90 \\ 0 & 0 & 0 & 0 & 0 & 1 & 20 \end{array}\right)$$

d. Solve the system of equations in (c).

We know that $t_2 = 1$, $t_4 = 10$, $t_5 = 90$, and $t_6 = 20$. We have from R_1 that $t_1 + \frac{1}{2} \cdot 20 = 15 \implies t_1 = 5$. From R_3, we have $t_3 - \frac{1}{2}t_5 - \frac{1}{2}t_6 = -11 \implies t_3 - 45 - 10 = -11 \implies t_3 = 44$.

Problem 2

Let b represent the bonuses paid to employees, in dollars. We have the following relationships given to us by the problem:

$$t_{\text{state}} = (103,000 - b) \cdot 0.05$$
$$t_{\text{federal}} = (103,000 - b - t_{\text{state}}) \cdot 0.4$$
$$b = (103,000 - t_{\text{state}} - t_{\text{federal}}) \cdot 0.05$$

For the convenience of notation, we will use s to represent the amount paid in state taxes (s_{state}) and f to represent the amount paid in federal taxes (s_{federal}). Let us now rearrange equations:

$$s = (103,000) - b) \cdot 0.05 \implies s = 5,150 - 0.05b \implies s + 0.05b = 5,150$$
$$f = (103,000 - b - s) \cdot 0.4 \implies f = 41,200 - 0.4b - 0.4s \implies 0.4s + f + 0.4b = 41,200$$
$$b = (103,000 - s - f) \cdot 0.05 \implies b = 5,150 - 0.05s - 0.05f \implies 0.05s + 0.05f + b = 5,150$$

We can now construct the following augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5150 \\ 0.4 & 1 & 0.4 & 41200 \\ 0.05 & 0.05 & 1 & 5150 \end{array}\right)$$

Let us eliminate the first column using $-\frac{R_1 - \frac{5}{2}R_2}{2.5} \to R_2$...

$$\left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5150 \\ 0 & -2.5 & -0.95 & -97850 \\ 0.05 & 0.05 & 1 & 5150 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5150 \\ 0 & 1 & 0.38 & 39140 \\ 0.05 & 0.05 & 1 & 5150 \end{array}\right)$$

...and $R_1 - 20 \cdot R_2 \to R_2$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5150 \\ 0 & 1 & 0.38 & 39140 \\ 0 & -1 & -19.95 & -97850 \end{array}\right)$$

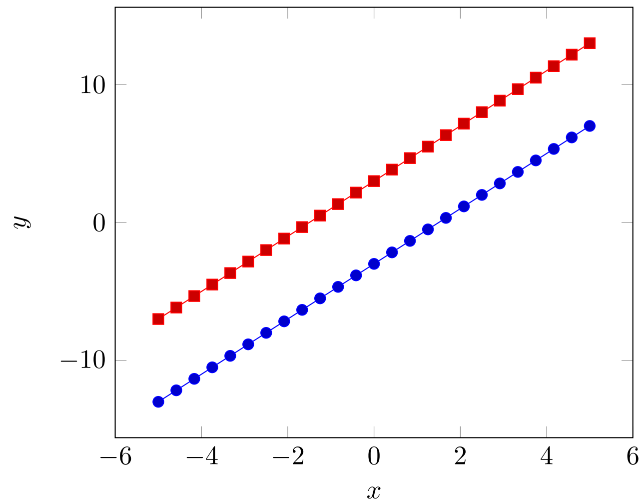
We can then clear the second column with $R_2 + R_3 \to R_3$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0.05 & 5150 \\ 0 & 1 & 0.38 & 39140 \\ 0 & 0 & -19.57 & -58710 \end{array}\right)$$

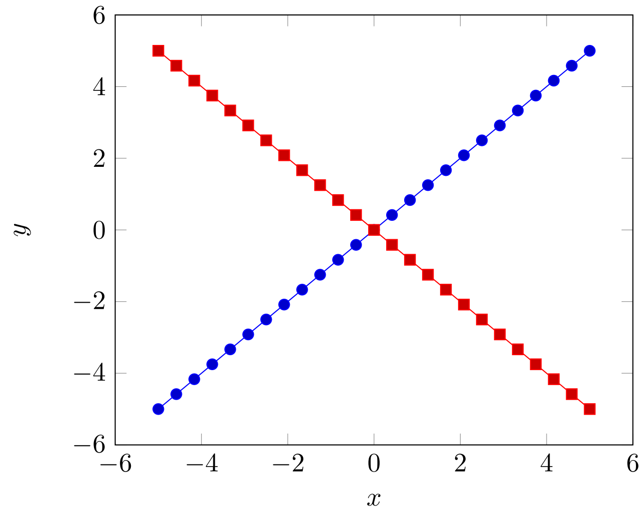
From this, we can compute that $-19.57b = -58710 \implies b = 3000$; $f + 0.38 \cdot 3000 = 39140 \implies 38000$; and $s + 0.05 \cdot 3000 = 5150 \implies 5000$. Therefore, **the company pays \$3000 in bonuses, \$5000 in state taxes, and \$38000 in federal taxes.**

Problem 3

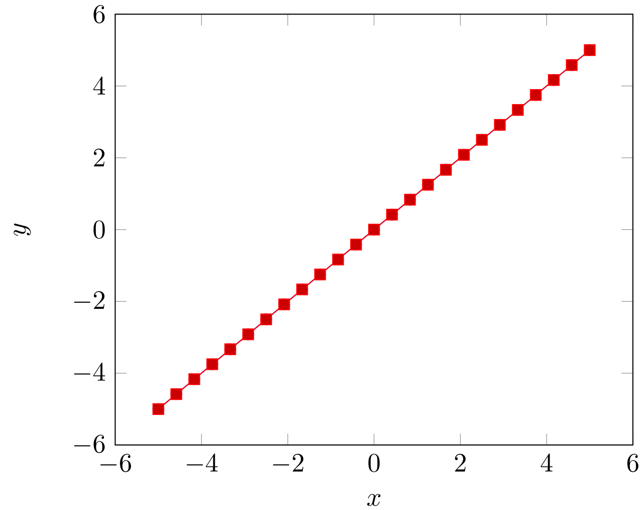
a. has no solution



b. has exactly one solution

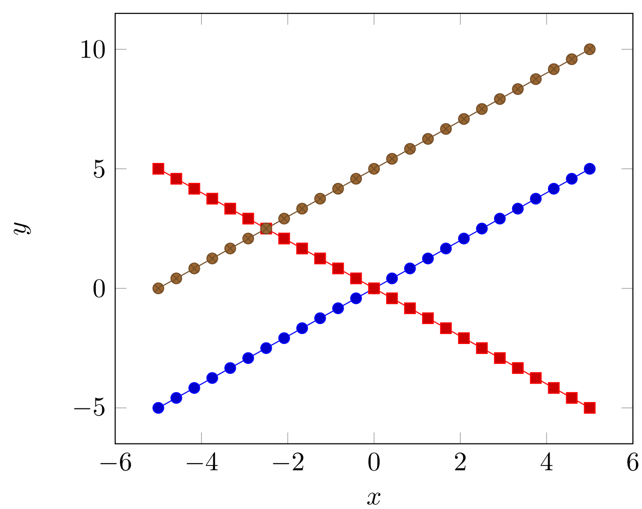


c. has infinitely many solutions



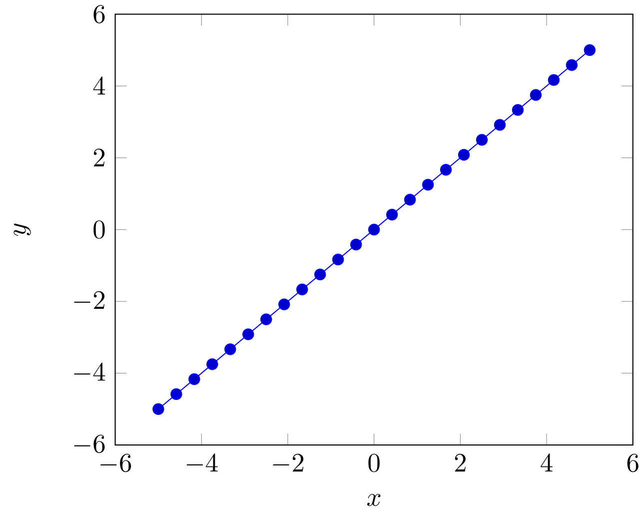
i. Add or remove equations in (b) to make an inconsistent system.

We can add another line which is parallel to an existing one; this makes it such that there is no solution to those two lines, and therefore no solution to the system overall.



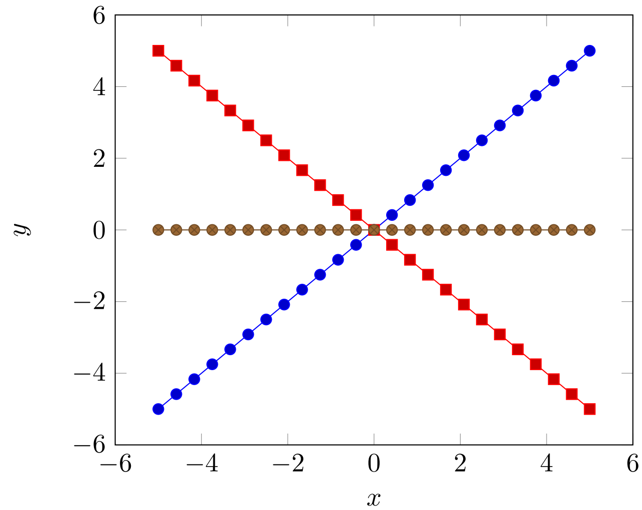
ii. Add or remove equations in (b) to create infinitely many solutions.

By removing one of the equations, there is now only one line; therefore there exist an infinite number of solutions which satisfy this single-equation system.



iii. Add or remove equations in (b) so that the solution space remains unchanged.

We can add an equation which crosses over the intersection of thew current system such that the overall resolution remains constant.



iv. Can you add or remove equations in (b) to change the unique solution you had to a different unique solution?

No, you can not. Say that an arbitrarily sized system of linear equations satisfies the constraint imposed in b) – namely, that there is one solution. If we remove any linear equation, the solution either remains the same or is no longer unique (i.e. there are an infinite number of solutions due to the presence of at least one under-specified dimension). If we add any linear equation, the solution either remains the same or the solution is lost.

Problem 4

a. What is the smallest number of equations you would need? Write down such a system.

You would need two equations. One system is $\{x = 2\}, \{y = 3\}$.

b. Can you add one more equation to the system in (a) so that the new system still has the unique solution $(2, 3)$*?*

Yes, you can add $y - 3 = \pi(x - 2)$.

c. What is the maximum number of distinct equations you can add to your system in (a) to still maintain the unique solution $(2, 3)$*?*

You can add an infinite number of distinct equations and still maintain the unique solution.

d. Is there a general form for the equations in (c)?

Yes, it is $y - 3 = m(x - 2)$, where $m \in \mathbb{R}$.

Problem 6

a. We want $f(x)$ to pass through the points $(-1, -1), (1, 2), (2, 1)$, and $(3, 5)$.

$$\begin{aligned}
-1 &= a_0 - a_1 + a_2(1) - a_3 \\
2 &= a_0 + a_1 + a_2 + a_3 \\
1 &= a_0 + 2a_1 + 4a_2 + 8a_3 \\
5 &= a_0 + 3a_1 + 9a_2 + 27a_3
\end{aligned}$$

b. We want $f(x)$ to pass through $(1, 0)$ with derivative $+2$ and $(2, 3)$ with derivative -1.

We have two positional constraints:

$$0 = a_0 + a_1 + a_2 + a_3$$

$$3 = a_0 + 2a_1 + 4a_2 + 8a_3$$

The derivative of $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ is $f'(x) = a_1 + 2a_2 x + 3a_3 x^2$. Given derivative requirements, we can add two additional equations:

$$2 = a_1 + 2a_2 + 3a_3$$

$$-1 = a_1 + 4a_2 + 12a_3$$

c. What if we had more than four points to consider? Fewer?

A polynomial with degree three has four coefficients; it can be expressed as a relationship between at four linear equations (as previously demonstrated). More generally speaking, a polynomial of degree n has $n + 1$ coefficients (unknowns) and requires $n + 1$ equations. If we had more than four points to consider, either a) there would be no solution because no degree-3 polynomial passes through the given set of points, or b) there will be a solution because the degree-3 polynomial specified by any size-4 subset of the set of all linear equations specifies the same polynomial. If we had fewer than four points, then the polynomial fit would be under-specified and there would be an infinite set of valid degree-three polynomials.

d. Suppose we know two points that $f(x)$ passes through and want to find two unknown constants that are in the formula for $f(x)$. Can we still use linear algebra if $f(x)$ is another kind of function such as $f(x) = a \sin x + b \cos x$? $f(x) = a \sin(bx)$? $f(x) = ae^{bx}$?

We can use linear algebra even for non-polynomial functions if the relation between the parameters is still linear. For instance, in $f(x) = a \sin x + b \cos x$, the relationship between the two parameters a and b are linearly related to each other. In both $f(x) = a \sin(bx)$ and $f(x) = ae^{bx}$, however, the relationship is nonlinear, and therefore we cannot reliably use linear algebra to solve for the parameters.