Optimum and Convergence Patterns in Power Towers

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1 Introduction

The monomial - which is the simplest form of the polynomial - and the exponential have a syntax similar to each other; the former can be expressed in simple form as x^a where as the latter is a^x. A natural morphological fusion of these two yields x^x. However, such a function does not seem to have the parametric controls that the polynomial and the exponential possess; hence, we may "dress" the function with parameters a and b to form the function $f(x) = ax^{bx}$. Two limitations must be placed on these parameters, though; it must be true that $a \neq 0$ and $b \neq 0$. If $a = 0$, the result is $f(x) = 0$. If $b = 0$, the result is $f(x) = a$. Neither of these cases, where $a = 0$ or $b = 0$, allow the function to have an extremum.

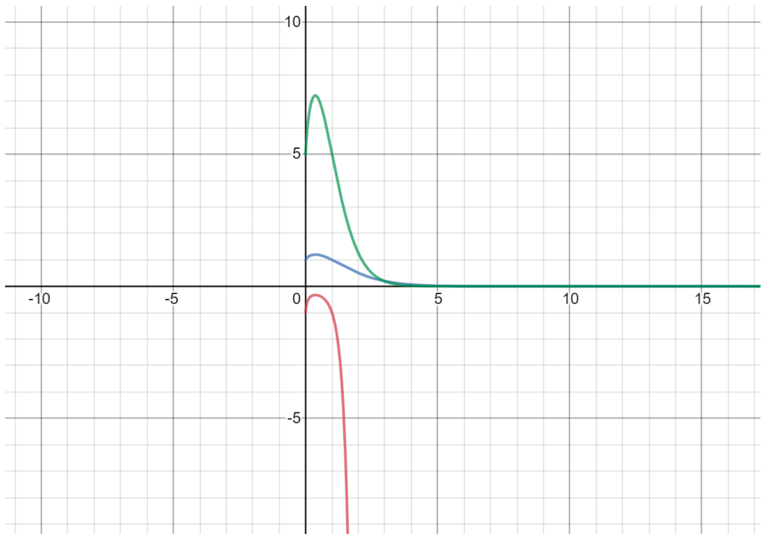

Figure 1.1: x^x in red, $x^{-0.5x}$ in blue, $5x^{-x}$ in green. Note that the values of a and b are arbitrary, and the values chosen here function to display the diversity in the possible function outputs.

As we observe many different values of a and b for variations of ax^{bx} and x^{bx} (we explore x^x as well, but it doesn't have alternate variations with a and b we can explore), we find two key observations:

1. The functions shown above seem to have only one extremum.
2. The functions' extremum can either be a minimum or a maximum – i.e., the function "points" upwards or downwards.

Hence, correspondingly in this paper, we seek to accomplish two goals:

1. Prove that there is only one extremum for any possible variations of the three functions we explore and find the value of this extremum.
2. Determine whether the extremum is a minimum or a maximum.

2 Deriving Solutions for Extrema in a Successive Manner

2.1 For $f(x) = x^x$

We can begin with the simplest version of the function $f(x) = ax^{bx}$, where $a = b = 1$. Let us take the derivative of $f(x)$, set the acquired derivative to zero to find the x-coordinate, and plug the x-value into the function in to find the y-value of the extrema.

$$\begin{aligned}
\frac{d}{dx}f(x) &= \frac{d}{dx}(x^x) \\
&= e^{x\ln(x)}\frac{d}{dx}(x\ln(x)) \\
&= e^{x\ln(x)}(\ln(x)+1) \\
&= x^x(\ln(x)+1) \\
0 &= x^x(\ln(x)+1)
\end{aligned}$$

In order for the above expression to be true, $(\ln(x)+1)$ must equal 0, as it is not possible for x^x to equal 0. Thus, to determine when the above equation stands true, we must determine when $(\ln(x)+1)$ equals 0.

$$\begin{aligned}
\ln(x)+1 &= 0 \\
\ln(x) &= -1 \\
x &= \frac{1}{e}
\end{aligned}$$

Plugging this value of x back into $f(x) = x^x$, we can solve for the y-coordinate of the extremum to be:

$$f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}}$$

So the location of the extrema is:

$$\left(\frac{1}{e}, \left(\frac{1}{e}\right)^{\frac{1}{e}}\right)$$

We can determine if this point is the minimum or maximum by taking the second derivative of $f(x)$.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{d}{dx}f(x)\right) &= \frac{d}{dx}(x^x(\ln(x)+1)) \\
&= \frac{d}{dx}(x^x(\ln(x)+1)) \\
&= \left(\frac{d}{dx}(x^x)\right)(\ln(x)+1) + (x^x)\frac{d}{dx}(\ln(x)+1)) \\
&= \left(x^x(\ln(x)+1)(\ln(x)+1) + \frac{1}{x}(x^x)\right) \\
&= \left(x^x(\ln(x)+1)^2 + x^{x-1}\right)
\end{aligned}$$

We that if $\frac{d}{dx}\left(\frac{d}{dx}f(x)\right)$ is positive, the extremum is a minimum, and if it is negative, the extremum is a maximum. So, solving for the sign of $\frac{d}{dx}\left(\frac{d}{dx}f(x)\right)$, we get:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{d}{dx}f\left(\frac{1}{e}\right)\right) &= \left(\frac{1}{e}\right)^{\frac{1}{e}}\left(\ln\left(\frac{1}{e}\right)+1\right)^2 + \left(\frac{1}{e}\right)^{\frac{1}{e}-1} \\
&= \left(\ln\left(\frac{1}{e}\right)+1\right)^2\left(\frac{1}{e}\right)^{\frac{1}{e}} + \left(\frac{1}{e}\right)^{\frac{1}{e}-1}
\end{aligned}$$

Because $\left(\ln\left(\frac{1}{e}\right)+1\right)^2$ is always equal to 0, the first term of the expression above - $\left(\ln\left(\frac{1}{e}\right)+1\right)^2\left(\frac{1}{e}\right)^{\frac{1}{e}}$ - is 0. The second term $\left(\frac{1}{e}\right)^{\frac{1}{e}-1}$ is positive, so this extremum is a minimum.

2.2 For $f(x) = x^{bx}$

Let's take another step towards fully "dressing" our function. For our purposes in this section, we can redefine $f(x)$ to be $f(x) = x^{bx}$. We can use the same method that we used in section 2.1 to find the extremum. First, lets take the derivative of $f(x)$:

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{d}{dx}\left(x^{bx}\right)\\ &= e^{bx\ln(x)}\frac{d}{dx}(bx\ln(x))\\ &= e^{bx\ln(x)}b(\ln(x)+1)\\ &= bx^{bx}(\ln(x)+1)\end{aligned}$$

Set the acquired derivative of $f(x)$ to 0 to calculate the x-coordinate of the extremum.

$$bx^{bx}(\ln(x)+1) = 0$$

Note that since bx^{bx} and $(\ln(x)+1)$ are multiplied together, only one of these two need to equal 0 in order for the equation we have set above to be true. Looking back to our introduction, we set the condition that b can never equal 0. This means that bx^{bx} can never equal 0, so $(\ln(x)+1)$ is the expression that has to equal 0.

$$\begin{aligned}\ln(x)+1 &= 0\\ \ln(x) &= -1\\ x &= \frac{1}{e}\end{aligned}$$

We now know the x-coordinate of the extremum. Now, we must plug in $\frac{1}{e}$ for x in $f(x)$, and solve for $f(\frac{1}{e})$ to obtain the y-coordinate.

$$f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{\frac{b}{e}}$$

We can't simplify our value of $f(\frac{1}{e})$ obtained above in any meaningful ways, so we know that the location of the extremum is:

$$\left(\frac{1}{e}, \left(\frac{1}{e}\right)^{\frac{b}{e}}\right)$$

Now, we must determine if this point is the minimum or maximum. To do so, we can take the second derivative of $f(x)$:

$$\begin{aligned}\frac{d}{dx}\left(\frac{d}{dx}f(x)\right) &= \frac{d}{dx}\left(bx^{bx}(\ln(x)+1)\right)\\ &= b\frac{d}{dx}\left(x^{bx}(\ln(x)+1)\right)\\ &= b\left(\frac{d}{dx}\left(x^{bx}\right)(\ln(x)+1) + \frac{d}{dx}\left(x^{bx}\right)(\ln(x)+1)\right)\\ &= b\left(bx^{bx}\left(\ln(x)+1\right)\left(\ln(x)+1\right) + \frac{1}{x}(x^{bx})\right)\\ &= b\left(bx^{bx}\left(\ln(x)+1\right)^2 + x^{bx-1}\right)\end{aligned}$$

Now that we have found the second derivative, we can determine if our extremum is a minimum or maximum. We know that the x coordinate of the extremum is $\frac{1}{e}$, so if $\frac{d}{dx}\left(\frac{d}{dx}f(x)\right)$ is positive, then the extremum is a minimum, and if $\frac{d}{dx}\left(\frac{d}{dx}f(x)\right)$ is negative, it is a maximum. Let's plug in our value of x into the second derivative:

$$\frac{d}{dx}\left(\frac{d}{dx}f\left(\frac{1}{e}\right)\right) = b\left(b\left(\frac{1}{e}\right)^{\frac{b}{e}}\left(\ln\left(\frac{1}{e}\right)+1\right)^2 + \left(\frac{1}{e}\right)^{\frac{b}{e}-1}\right)$$
$$= \left(\ln\left(\frac{1}{e}\right)+1\right)^2\left(\frac{1}{e}\right)^{\frac{b}{e}} b^2 + \left(\frac{1}{e}\right)^{\frac{b}{e}-1} b$$

We know that for any value of b, $\left(\ln\left(\frac{1}{e}\right)+1\right)^2\left(\frac{1}{e}\right)^{\left(\frac{1}{e}\right)b} b^2$ is equal to 0 because $\left(\ln\left(\frac{1}{e}\right)+1\right)^2$ is always equal to 0.

Additionally, $\left(\frac{1}{e}\right)^{\left(\frac{1}{e}\right)b-1}$ is always positive, but it is being multiplied by b. This means that when $b > 0$, $\left(\frac{1}{e}\right)^{\left(\frac{1}{e}\right)b-1} b$ - and by extension, the entire second derivative - is positive, and when $b < 0$, $\left(\frac{1}{e}\right)^{\left(\frac{1}{e}\right)b-1} b$ - and again, the entire second derivative - is negative.

This means that when $b > 0$, our extremum is a minimum, and when $b < 0$, our extremum is a maximum.

2.3 For $f(x) = ax^{bx}$

Building off of the prior sections, we can finally approach the fully "dressed" form of $f(x)$ mentioned in the introduction. The methods that we use are virtually identical to the sections that came before as well. Like we did previously, we can begin finding the extremum by first finding the derivative of $f(x) = ax^{bx}$.

$$\begin{aligned}
f(x) &= \frac{d}{dx}\left(ax^{bx}\right) \\
&= a\frac{d}{dx}\left(e^{bx\ln(x)}\right) \\
&= e^{bx\ln(x)}\frac{d}{dx}\left(bx\ln(x)\right) \\
&= ae^{bx\ln(x)}b\left(\ln(x)+1\right) \\
&= abx^{bx}\left(\ln(x)+1\right)
\end{aligned}$$

We can then set this derivative to 0 to calculate the x-coordinate of the extremum.

$$abx^{bx}\left(\ln(x)+1\right) = 0$$

According to the equation, either a, b, or $(\ln(x)+1)$ must be equal to 0 in order to make it true. However, because we specifically defined that a and b cannot be 0 in our introduction, $(\ln(x)+1)$ is the only expression that can equal 0.

$$\begin{aligned}
\ln(x)+1 &= 0 \\
\ln(x) &= -1 \\
x &= \frac{1}{e}
\end{aligned}$$

We can plug the x-coordinate into the original function $f(x) = ax^{bx}$ to get our y-coordinate in terms of a and b.

$$f\left(\frac{1}{e}\right) = a\left(\frac{1}{e}\right)^{\frac{b}{e}}$$

Now we know both coordinate values for the function $f(x)$. Therefore, our point in terms of a and b is

$$\left(\frac{1}{e}, a\left(\frac{1}{e}\right)^{\frac{b}{e}}\right)$$

As for deciding whether $f(x)$ has a minimum or maximum, we can use similar logic from section 2.2. Taking the second derivative of $f(x)$,

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{d}{dx}f(x)\right) &= \frac{d}{dx}\left(abx^{bx}\left(\ln(x)+1\right)\right)\\
&= ab\frac{d}{dx}\left(x^{bx}\left(\ln(x)+1\right)\right)\\
&= ab\left(\frac{d}{dx}\left(x^{bx}\right)\left(\ln(x)+1\right)+\frac{d}{dx}\left(\ln(x)+1\right)x^{bx}\right)\\
&= ab\left(bx^{bx}\left(\ln(x)+1\right)\left(\ln(x)+1\right)+\frac{1}{x}x^{bx}\right)\\
&= ab\left(bx^{bx}\left(\ln(x)+1\right)^2+x^{bx-1}\right)
\end{aligned}
$$

We can then take this second derivative and plug in $\frac{1}{e}$ for x, with the knowledge that a positive expression leads to a minimum point in $f(x)$, while a negative leads to a maximum.

$$\frac{d}{x}\left(\frac{d}{x}f\left(\frac{1}{e}\right)\right)=ab\left(b\left(\frac{1}{e}\right)^{\frac{b}{e}}\left(\ln\left(\frac{1}{e}\right)+1\right)^2+\left(\frac{1}{e}\right)^{\frac{b}{e}-1}\right)$$

We reasoned that $(\ln(x)+1)$ must be equal to 0 when $\frac{1}{e}=0$. Therefore, the expression simplifies to

$$ab\left(\left(\frac{1}{e}\right)^{\frac{b}{e}-1}\right)$$

The expression inside the parentheses must always be positive, because $\frac{1}{e}$ is positive. In that case, if b is positive, the only way for the second derivative to be negative would be if a was negative. Similarly, if b is negative, the entire second derivative would be negative unless a is negative. With this logic, we can confirm that when $ab>0$, $f(x)$ has a minimum, and when $ab<0$, $f(x)$ has a maximum.

2.4 Analysis

Through the usage of derivatives, we were able to find the point of the extremum for $f(x)=ax^{bx}$, which is shown below.

$$\left(\frac{1}{e},a\left(\frac{1}{e}\right)^{\frac{b}{e}}\right)$$

We were able to determine whether the extremum is a minimum or a maximum by finding the second derivative of $f(x)$. If $ab>0$, $f(x)$ will have a minimum, while if $ab<0$, $f(x)$ will have a maximum. This is shown in context through Figure 2.1.

Additionally, we were able to find the extrema of less fully dressed versions of $f(x)=ax^{bx}$, which helped us build up to the fully dressed version. When $f(x)=x^x$, there was a minimum at $\left(\frac{1}{e},\left(\frac{1}{e}\right)^{\frac{1}{e}}\right)$. In addition, when $f(x)=x^{bx}$, there was an extrema at $\left(\frac{1}{e},\left(\frac{1}{e}\right)^{\frac{b}{e}}\right)$, which is a minimum when $b>0$ and a maximum when $b<0$.

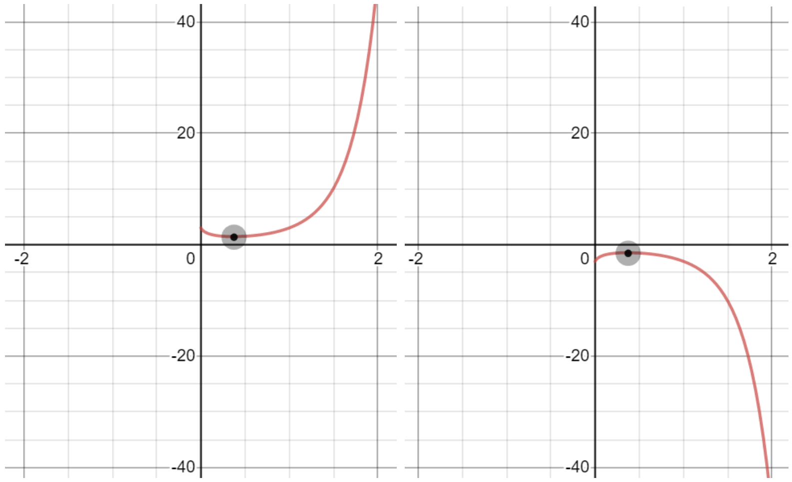

Figure 2.1: The graph of $f(x) = ax^{bx}$ when $a = 3$ and $b = 2$ (left) and when $a = -3$ and $b = 2$ (right), with the extremum shown as the point in black.

From Figure 2.1, we can verify our conclusion about the sign of ab and how it shows whether the graph has a minimum or maximum.

Interestingly, when graphing $abx^{bx}(\ln(x) + 1) = 0$ on Desmos for particular values of a and b on Desmos, it displays two solutions, as shown in Figure 2.2. However, because of our calculations, we know for a fact that the only solution is at $x = \frac{1}{e}$.

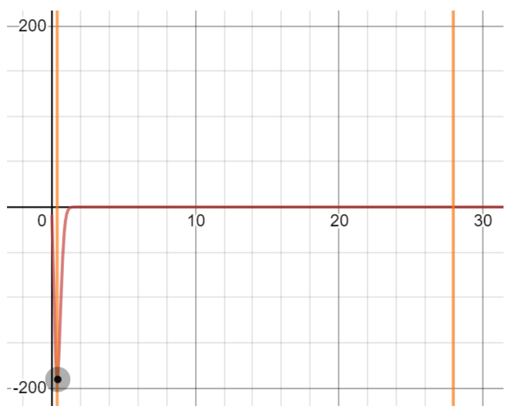

Figure 2.2: The graph of $f(x) = ax^{bx}$ is in red, where $a = -10$ and $b = -8$. The graph of $\frac{d}{dx}(ax^{bx}) = 0$ is in orange, with the extremum shown as the point in black.

Desmos's rendering of the graph displays $\frac{d}{dx}(ax^{bx}) = 0$ more than one instance where the derivative of $f(x)$ equals 0. This is likely a Desmos error in dealing with large numbers, and has the consequence of tricking unsuspecting students doing a math project that there are in fact *two* solutions for the x-coordinate of the extremum. Despite this, since we have proven algebraically that the only possible extremum occurs at $x = \frac{1}{e}$, this is no longer an issue.

3 Extension

In this paper, we explored the minimum and maximum of ax^{bx}. Naturally, this leads to questioning of the behavior of further exponentiation, leading to $ax^{bx^{cx}}$, $ax^{bx^{cx^{dx}}}$, and so on and so forth. To simplify this expression as to make it more manageable, let us assume the case where $a = b = c = d = \ldots$, yielding $ax^{ax^{\cdots}}$. In order to refer to this power tower in a more compact manner, we can define $f_n(x)$ to be used to refer to the power tower defined by ax exponentiated to the ax power n times. For instance, $ax^{ax^{ax}}$ can be written as $f_2(x)$, and ax^{ax} as $f_1(x)$. Building upon this definition, for this paper we define *odd* functions as functions $f_n(x)$ such that $\mathrm{mod}(n, 2) = 1$ – for instance, $f_1(x)$, $f_3(x)$, etc. – and *even* functions correspondingly to be functions $f_n(x)$ such that $\mathrm{mod}(n, 2) = 0$ – for instance, $f_2(x)$, $f_4(x)$, etc. Note that this is different from definitions of even and odd functions that refer to $f(x) = f(-x)$ and $f(-x) = -f(x)$, which will not be referenced at all in this section.

When $a < 0$, we observe an interesting pattern – all "odd" iterations appear to converge to $y = 0$, whereas all "even" iterations appear to converge to $y = a$ when graphed.

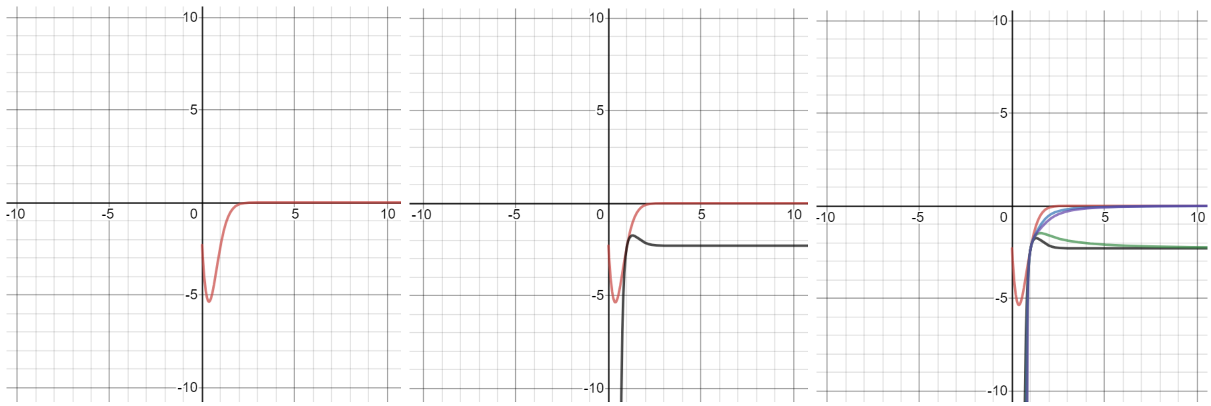

Figure 3.1: "Even" iterations seem to converge to $y = a$, where "odd" iterations seem to converge to $y = 0$. $f_1(x)$ in red, $f_2(x)$ in black, $f_3(x)$ in blue, $f_4(x)$ in green, $f_5(x)$ in purple.

Our extension seeks to prove that this is the case.

3.1 The Rationale for Exploring $a < 0$

First, it is important to explore the initially seemingly arbitrary restriction of $a < 0$, and why this analysis would explore only values of a such that $a < 0$. To begin with, let us define $b = |a|$. Note that this definition of b differs from the notation used in the previous section of the paper, in which b played the role of ax^{bx}. This new definition of b results in $-b = a$.

The reason why we define $b = |a|$ (for the purpose of having $-b = a$) is because a already has a negative value, so when we put a negative sign in front of a, it will always result in something positive. However, the negative signs (which we would have been able to manipulate if we had picked an actual numerical value of a in the domain $a < 0$) are important to help us manipulate any $f_n(x)$ when $a < 0$ in a way that we can prove what $f_n(x)$ converges to. Thus, by setting $b = |a|$, we are able to manipulate the negative signs in the way that are needed to prove what the functions converge to.

The function $ax^{ax^{ax^{\cdots}}}$, for positive values of a, increases to ∞ for any value of a under this condition. If $ax > 1$, then the power tower must converge to infinity. Given that $a > 0$ and $x > 0$, the graph of ax must be larger than 1 at some value of x for any value of a, since a acts as the slope of this linear equation. Furthermore, the linear nature of ax means that as one follows the graph as x increases, once $ax > 1$, this inequality remains true. Thus, as $x \to \infty$, ax must be larger than 1. The power tower hence always approaches ∞ as $x \to \infty$ for $a > 0$.

On the other hand, using $b = -a$, we can write $ax^{ax^{ax^{\cdots}}}$ as $-bx^{-bx^{-bx^{\cdots}}}$. If we consider a simple example of $f_2(x) = -bx^{-bx^{-bx}}$, we can rewrite it as $-bx^{-\frac{b}{x^{bx}}}$. We can then write this resulting exponent as $-\frac{b}{x^{\frac{b}{x^{bx}}}}$. By using b, we can explore interesting resulting fractions that "nest" themselves in a way that begs further exploration.

3.2 When $a < 0$

Let's confirm mathematically what values of a which are less than 0 converge to. Specifically, we are looking to confirm that even values of n converge to $y = a$ and odd values of n converge to $y = 0$.

We know that $b = |a|$. This means that $-b = a$. When we replace all the a in $f_2(x)$ with $-b$, we are left with $f_2(x) = -bx^{-bx^{-bx}}$. Let's change the form of $f_2(x)$:

$$\begin{aligned} f_2(x) &= -bx^{-bx^{-bx}} \\ &= -\frac{b}{x^{bx^{-bx}}} \end{aligned}$$

Let's start with proving that as x approaches ∞, $f_2(x)$ approaches a.

First, we can prove that bx^{-bx}, which is the power that x in the denominator is taken to, approaches 0 as x approaches ∞. This will lead us to prove that $x^{bx^{-bx}}$ approaches 1 as x approaches ∞.

We know that bx^{-bx} simplifies to $\frac{b}{x^{bx}}$. As x grows larger, the value of the expression's denominator gets larger and larger, while the numerator stays the same. As the denominator of $\frac{b}{x^{bx}}$ grows larger and larger as x increases, the output grows closer and closer to 0, because the number of times that b is being divided increases. This cycle always increases as x increases, so as x approaches ∞, the output approaches 0. Thus, we can represent this as:

$$\lim_{x\to\infty} y = \frac{b}{\infty}$$

Here, b is a constant, and any constant divided by something in magnitude near ∞ is going to be very close to 0.

Additionally, this means that when we take x to the power of bx^{-bx}, as the value of x approaches ∞, the power that the base of the exponent is being taken to approaches 0. This means that x is taken to the power of a value that approaches 0. When anything is taken to the power of 0, it results in an output of 1. Thus, $x^{bx^{-bx}}$ approaches 1 as x approaches ∞. Since $-b$ is divided by $x^{bx^{-bx}}$, $-b$ ends up being divided by something that approaches 1 as x approaches ∞, so $f_2(x)$ approaches $-b$ (which means that it approaches a, because $-b = a$) as x approaches ∞.

We can extend this property, of the function output approaching a as x approaches ∞ to all $f_n(x)$ when n is even. In the process of doing this, we can prove that for all odd n, the function output approaches 0 as x approaches ∞.

Here, we can see with $f_2(x)$:

$$f_2(x) = -\frac{b}{x^{bx^{-bx}}}$$

Take the limit of $f_2(x)$:

$$\lim_{x\to\infty} = \frac{\lim_{x\to\infty} -b}{\lim_{x\to\infty}\left(x^{bx^{-bx}}\right)}$$

We already know that the limit of the denominator is 1 because $-bx^{-bx}$ approaches $\frac{b}{\infty} \to 0$ as x approaches ∞. Thus, the entire function approaches a as x approaches ∞. However, when we set $-bx^{-bx}$ to the power of $-bx$ (which is transforming $f_2(x)$ to $f_3(x)$, as $f_3(x)$ is taken to the power of an extra $-bx$ compared to $f_2(x)$), we are setting something that approaches $\frac{b}{\infty}$ as x approaches ∞ to $-bx$.

Let's observe a $f_3(x)$; here, we can see:

$$f_3(x) = \frac{-b}{x^{bx^{-bx^{-bx}}}}$$

$$f_3(x) = \frac{-b}{x^{bx^{\frac{-b}{x^{bx}}}}}$$

$$f_3(x) = \frac{-b}{x^{\frac{b}{x^{\frac{b}{x^{bx}}}}}}$$

Now, take the limit of $f_3(x)$ as x approaches ∞

$$\lim_{x\to\infty} f_3(x) = \lim_{x\to\infty}\left(\frac{-b}{x^{\frac{b}{x^{\frac{b}{x^{bx}}}}}}\right)$$

As we have proven previously, $\frac{b}{x^{bx}}$ approaches 0 as x approaches ∞. Thus, we can simplify the above limit to:

$$\lim_{x\to\infty} f_3(x) = \lim_{x\to\infty}\left(\frac{-b}{x^{\frac{b}{x^0}}}\right)$$
$$\lim_{x\to\infty} f_3(x) = \lim_{x\to\infty}\left(\frac{-b}{x^{\frac{b}{1}}}\right)$$
$$\lim_{x\to\infty} f_3(x) = \lim_{x\to\infty}\left(\frac{-b}{x^b}\right)$$
$$\lim_{x\to\infty} f_3(x) = \left(\frac{-b}{\infty^b}\right)$$
$$\lim_{x\to\infty} f_3(x) = \frac{-b}{\infty}$$

Now, we know that $f_3(x)$'s denominator approaches ∞ as x approaches ∞.

Thus, when we take $\frac{-b}{1}$ to the power of $-bx$ as x approaches ∞, you are left with $\frac{-b}{\infty}$.

Now, we can prove that when you take $\frac{-b}{\infty}$ to the power of $-bx$, you are left with $\frac{-b}{1}$. Earlier, we proved that $f_2(x)$ approaches a as x approaches ∞. The only difference between $f_2(x)$ and $f_1(x)$ is that $f_2(x)$ is taken to the power of $-bx$ one time more than $f_1(x)$. Thus, there is only one possible reason that the lines converge to two different values. If we can show that $f_2(x)$ approaches $\frac{-b}{1}$ as x approaches ∞, then we can prove that when you take $\frac{-b}{\infty}$ to the power of $-bx$, you are left with $\frac{-b}{1}$.

When $n = 1$, here is $f_n(x)$:

$$f_1(x) = -bx^{-bx}$$
$$f_1(x) = \frac{-b}{x^{bx}}$$

Now, find the limit of $f_1(x)$ as x approaches ∞:

$$\lim_{x\to\infty} f_1(x) = \frac{\lim_{x\to\infty}(-b)}{\lim_{x\to\infty}(x^{bx})}$$
$$\lim_{x\to\infty} f_1(x) = \frac{-b}{\infty^{b\infty}}$$
$$\lim_{x\to\infty} f_1(x) = \frac{-b}{\infty}$$

As we mentioned earlier, any constant divided something in magnitude near ∞ is going to be very close to 0. So, the output of $\frac{-b}{\infty}$ approaches 0 as x approaches ∞.

To summarize, when something that approaches $\frac{-b}{\infty}$ is taken to the power of $-bx$, we are left with something that approaches $\frac{-b}{1}$ as x approaches ∞ and when $\frac{-b}{1}$ is taken to the power of $-bx$, we are left with $\frac{-b}{\infty}$ as x approaches ∞.

We know this because $f_1(x)$ approaches $\frac{-b}{\infty}$ as x approaches ∞. When $f_1(x)$ is taken to the power of $-bx$, it turns into $f_2(x)$. $f_2(x)$ approaches $\frac{-b}{1}$ as x approaches ∞. However, something that approaches $\frac{-b}{1}$ approaches $\frac{-b}{\infty}$ when it is taken to the power of $-bx$. We know this because $f_3(x)$, which is raised to the power of $-bx$ one time more than $f_2(x)$, approaches $\frac{-b}{\infty}$.

Thus, we have proved that when n is odd, the output of $f_n(x)$ approaches 0 as x approaches ∞. It is also proved that when n is even, the output of $f_n(x)$ approaches a as x approaches ∞.

3.3 Analysis

In the extension, we were able to prove that all odd power towers converged to $y = 0$, whereas all even ones converged to $y = a$, under the condition that $a < 0$. One requires such a framework to think about this problem rigorously; building upon the fundamental pillars derived in that framework, one can arrive at an inductive thinking paradigm that gets at the heart of this phenomena.

We have that $f_1(x) = -bx^{-bx} = -\frac{b}{x^{bx}}$, which was established to converge to zero. This makes sense, as the denominator, x^{bx}, grows much larger than the numerator, b.

Next, we have that $f_2(x) = -bx^{-bx^{-bx}}$. Alternatively, we can write this as $f_2(x) = -bx^{f_1(x)}$. It logically follows that we can define $f_n(x) = -bx^{f_{n-1}(x)}$. If we are to find the convergence of $f_2(x)$ as $x \to \infty$, we can substitute $f_1(x)$ with 0, the previously found convergence value; the resulting expression is $f_2(x) = -bx^0$. As $x^0 = 1$ for any x, $f_2(x)$ simplifies to $-b$, which is a. We have thus established that when a power tower converging to $y = 0$ is "stacked", the new power tower converges to $y = a$.

Lastly, to come "full circle" from odd $\to$ even $\to$ odd, we have that $f_3(x) = -bx^{f_2(x)}$. As $f_2(x)$ approaches $-b$, we can substitute to find that $f_3(x)$ converges to $-bx^{-b} \to -\frac{b}{x^b}$, which, as $x \to \infty$, approaches 0 – using the same logic that the denominator "outpaces" the numerator in growth with respect to the increase in the value of x. This move establishes that when a power tower converging to $y = a$ is "stacked", the new power tower converges to $y = 0$.

This formal statement of $f_n(x) = -bx^{f_{n-1}(x)}$ is an important tool that can be achieved after formulating the approach in 3.2, and allows us to understand this power tower's unique convergence pattern in a clean and elegant way.

4 Further Inquiry

In the extension question, we observed the pattern of "even" and "odd" stacking of power towers – $ax^{ax^{ax^{\cdots}}}$. We prove that convergence to the line $y = 0$ or $y = a$ as $x \to \infty$ acts as a test for whether the power tower has an even or odd number of exponentiated ax terms.

The next step, then, is to determine convergence for *infinite* power towers when $a < 0$. Given that it seems for all power towers with the condition $a < 0$, the tower converges to a certain value as $x \to \infty$, so it would seem reasonable that an infinite power tower would similarly converge to some value.

In the case where this is true, it would be interesting to see whether ∞ behaves as an even or odd power tower, if it converges to $y = a$ or $y = 0$ at all. This could have interesting applications of viewing ∞ as "even" or "odd", however ridiculous that may seem at face value. Alternatively, there may be another satisfying result, like an infinite power tower converging to $y = \frac{a}{2}$ or $y = \sqrt{|a|}$, which suggests ∞ is "somewhere between". Alternatively, it may be that while any finite power tower converges, an infinite power tower does not.