Homework 8

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Problem 17.6

Problem: Shirley is on a ferris wheel which spins at the rate of 3.2 revolutions per minute. The wheel has a radius of 45 feet, and the center of the wheel is 59 feet above the ground. After the wheel starts moving, Shirley takes 16 seconds to reach the top of the wheel. How high above the ground is she when the wheel has been moving for 9 minutes?

Answer: Let us impose a coordinate system such that the center of the wheel is located at $(0, 59)$. The Ferris wheel spins a 3.2 revolutions per minute, which means it travels $6.4\pi t$ radians for t minutes. The wheel has a radius of 45 feet, meaning it has a circumference of $90\pi \approx 282.7433$ feet. This means that a location on the edge of the Ferris wheel moves $3.2 \times 90\pi = 288\pi$ feet per minute. Because Shirley takes 16 seconds to reach the top of the wheel from her starting place, she is $288\pi \times \frac{16}{60} = \frac{384\pi}{5} \approx 241.27$ feet from the top. This is equivalent to being $90\pi - \frac{384\pi}{5} \approx 41.47$ feet from the top, although we are measuring from the other direction. Alternatively, Shirley is $\frac{90\pi - \frac{384\pi}{5}}{90\pi} \cdot 2\pi = \frac{22\pi}{75}$ radians from the top, or $\frac{22\pi}{75} + \frac{\pi}{2} = \frac{119\pi}{150}$ radians from the x-axis. We can thus write Shirley's x and y location at time t in minutes as

$$x(t) = 45\cos\left(6.4\pi t + \frac{119\pi}{150}\right)$$

$$y(t) = 45\sin\left(6.4\pi t + \frac{119\pi}{150}\right) + 59$$

It should be noted that although the problem does not give us the direction Shirley travels in, it is arbitrary. After 9 minutes, Shirley is $y(9) \approx 101.4969$ feet above the ground.

Problem 17.7

Problem: The top of the Boulder Dam has an angle of elevation of 1.2 radians from a point on the Colorado River. Measuring the angle of elevation to the top of the dam from a point 155 feet farther down river is 0.9 radians; assume the two angle measurements are taken at the same elevation above sea level. How high is the dam?

Solution: Let h be the height of the dam, in feet. We can construct two equations based on the information the problem gives us:

$$\tan(1.2) = \frac{h}{a}$$

$$\tan(0.9) = \frac{h}{a + 155}$$

We have two unknowns and two equations. We can solve for h and a by isolating h on one side in both equations and setting the other side of both equations equal to each other.

$$a\tan(1.2) = (a + 155)\tan(0.9) \implies \tan(1.2)a = \tan(0.9)a + 155\tan(0.9) \implies a = \frac{155\tan(0.9)}{\tan(1.2) - \tan(0.9)}$$

We then have the height of the dam to be $h = \tan(1.2)\left(\frac{155\tan(0.9)}{\tan(1.2)-\tan(0.9)}\right)$, or about 382.93 feet. Dam, that's one tall dam!

Problem 17.8

Problem: A radio station obtains a permit to increase the height of their radio tower on Queen Anne Hill by no more than 100 feet. You are the head of the Queen Anne Community Group and one of your team members asks you to make sure that the radio station does not exceed the limits of the permit. After finding a relatively flat area nearby the tower (not necessarily the same altitude as the bottom of the tower), and standing some unknown distance away from the tower, you make three measurements all at the same height above sea level. You observe that the top of the old tower makes an angle of 39° above level. You move 110 feet away from the original measurement and observe that the old top of the tower now makes an angle of 34° above level. Finally, after the new construction is complete, you observe that the new top of the tower, from the same point as the second measurement was made, makes an angle of 40° above the horizontal. All three measurements are made at the same height above sea level and are in line with the tower. Find the height of the addition to the tower, to the nearest foot.

Answer: Let h_b be the height of the tower before the addition. Let h be the total height of the tower, after the addition. The height of the addition is thus $h - h - b$. Let d be the distance the first measurement was taken from the station. The distance of the second measurement from the station is thus $d + 110$. We have the following relationships:

$$\tan(39^\circ) = \frac{h_b}{d}$$

$$\tan(34^\circ) = \frac{h_b}{d+110}$$

$$\tan(40^\circ) = \frac{h}{d+110}$$

Rearranging the first and second equations, we have that

$$d\tan(39^\circ) = (d+110)\tan(34^\circ) \implies d = \frac{110\tan(34^\circ)}{\tan(39^\circ) - \tan(34^\circ)}$$

We can use this to find h_b and h:

$$h_b = \tan(39^\circ)\left(\frac{110\tan(34^\circ)}{\tan(39^\circ) - \tan(34^\circ)}\right) = \frac{110\tan(39^\circ)\tan(34^\circ)}{\tan(39^\circ) - \tan(34^\circ)}$$

$$h = \tan(40^\circ)\left(\frac{110\tan(34^\circ)}{\tan(39^\circ) - \tan(34^\circ)} + 110\right)$$

We can now find the height of the addition:

$$h - h_b = \left(\tan(40^\circ)\left(\frac{110\tan(34^\circ)}{\tan(39^\circ) - \tan(34^\circ)} + 110\right)\right) - \left(\frac{110\tan(39^\circ)\tan(34^\circ)}{\tan(39^\circ) - \tan(34^\circ)}\right) \approx 108.37$$

The addition to the tower is thus about 108 feet tall.

Additional Problem 1

Problem: Solve the equation $\cos(2\theta + 1) = 0.5$. Then, open the "Genetic Optimization" tool (linked in the assignment description) and use it to find solutions to the equation. Which solutions does it find? How is the result affected by changing the parameters to the simulation?

Answer: We can solve the equation as follows:

$$\cos(2\theta+1) = 0.5$$

$$2\theta + 1 = \arccos 0.5 + 2\pi n \text{ or } \pi - \arccos 0.5 + 2\pi n$$

$$\theta = \frac{\arccos 0.5 + 2\pi n - 1}{2} \text{ or } \frac{\pi - \arccos 0.5 + 2\pi n - 1}{2}$$

We want to find solutions based on the function $\cos(2\theta+1)$ by measuring the difference between $\cos(2\theta+1)$ and 0.5. We can write this as $|0.5 - (\cos(2\theta+1))|$. We can use squaring rather than absolute value to express the same idea, forming $(0.5 - (\cos(2\theta+1)))^2$. Because the optimization tool finds maxima rather than minima, we can negate the fitness function: $-(0.5 - \cos(2\theta+1))^2$.

With a step size on its defaults at between 0 and 0.1, as long as the population is sufficiently high (larger than a quantity like one between 1 and 5-ish where random initialization can play into which solutions are found), the simulation appears to find solutions corresponding to $\frac{\arccos 0.5 + 2\pi n - 1}{2}$ and $\frac{\pi - \arccos 0.5 + 2\pi n - 1}{2}$, with $n = 0$ for both. When the step size maximum increases to 2, the simulation is able to find four solutions, corresponding to the principle and symmetric solutions where $n = 0, 1$. When the maximum step size increased to 10, the program found many more solutions to count (for reference, though, the window's x-range needed to be expanded to $(-30, 30)$ to accommodate all the solutions found.

Very interestingly, setting the minimum step size to 0.2 and the maximum step size to 2 allowed the simulation to discover two more solutions than when the maximum step size was the same but when the minimum step size was 0. It appears that the minimum step size prevented easily / immediately found solutions from being "too good" as to eliminate points discovering neighboring maxima.

Additional Problem 2

Problem: Solve the equation $3\tan(6\theta - 1) = 7$. Then, use the "Genetic Optimization" tool to find solutions to the equation. Which solutions does it find? How is the result affected by changing the parameters to the simulation?

Solution: We can solve the equation as follows:

$$\begin{aligned}3\tan(6\theta - 1) &= 7\\ \tan(6\theta - 1) &= \frac{7}{3}\\ 6\theta - 1 &= \arctan\left(\frac{7}{3}\right) + 2\pi n \text{ or } \pi - \arctan\left(\frac{7}{3}\right) + 2\pi n\\ \theta &= \frac{\arctan\left(\frac{7}{3}\right) + 2\pi n + 1}{6} \text{ or } \frac{\pi - \arctan\left(\frac{7}{3}\right) + 2\pi n + 1}{6}\end{aligned}$$

We want to find solutions based on the function $3\tan(6\theta - 1)$ by measuring the difference between $3\tan(6\theta - 1)$ and 7. Using the same logic as in additional problem 1, our objective function is thus

$$-\left(7 - 3\tan(6\theta - 1)\right)^2$$

With a step size between 0 and 0.1, the program finds four solutions corresponding to $n = -1$ and $n = 0$ for the principal and symmetric solutions. These four solutions that touch $x = 0$ occur between $x = -1$ and $x = 1$, which makes sense given the initialization domain. When the maximum step size increased to 1, the algorithm finds 10 solutions. Sometimes, solutions "disappeared" and then "reappeared", probably because a step size of 1 can carry an "agent" "further" in terms of the possible solution space compared to the function in additional problem 1.

Additional Problem 3

Problem: Gina is experimenting with the force of friction on curling ice. She places a broom at various angles θ from the vertical (see diagram) and leans her full weight (130 lb.) along the shaft. She then has an apparatus pull the head of the broom in front of her, to see how much force it takes to move it. Use the "Genetic Modeling" tool to find a model for the amount of force required when Gina leans at an angle of $x°$. Justify any decisions you make. What does the end result tell you about the situation?

Solution: The data seems to decrease as the angle increases. We can begin by modeling the data with a line $y = ax+b$. With a step size of 0.1 and setting at 500 generations per step, at about 6000 generations the algorithm converges to the model $y = -0.76x + 76.09$ with an RMSE of 6.38, which isn't bad at all.

We can do better, though; there seems to be a a tilt to the data such that for larger values of x the force required decreases by a larger amount. We can use a quadratic model $y = ax^2 + bx + c$. With a maximum step size of 0.1, the algorithm seems to converge to a local optima with an RMSE of 30.24. By beginning with meaningfully selected initialization values of a at -30 (the absolute value 30 itself is not relevant, but this is meant to guide the algorithm directly towards negative slopes) and c at 68.01 (the output when $x = 0$). The step size was set to 10 for all parameters and progressively decreased, yielding the model $y = -0.0069x^2 - 0.1959x + 70.7622$ with RMSE 1.6442, which is much better than the linear model.

The down-sloping side of the sinusoidal function can also be utilized. In this case, the model takes the form of $a\sin(b(x - c)) + d$. Similarly to with the quadratic function, by setting reasonable initial parameters, taking care with setting a low period so the model does not come to a sinusoidal with an arbitrarily low period that can pass through every point, the best sinusoidal model yields $-52.51\sin(-0.02(x + 74.64)) + 15.55$ with an even lower RMSE of 0.6562.

Both the quadratic and the sinusoidal function fit well, although the sinusoidal function fits even better. It seems that the relationship between x and the force required is "sinusoidal" not in the sense of it repeating, but it having a sinusoidal shape of descent.

Additional Problem 4

Context: Consider a very simple learning problem, where a program is trying to "learn" the best square in the table below.

Part A Problem: Suppose that the program maintains a population of just one candidate, and in each generation it chooses a new candidate by moving exactly one square either up, down, left, or right with equal likelihood. If the new candidate is better than the old one, it switches. If the program starts at $a = b = 1$, what's the probability that it will find the true optimum solution?

Part A Answer: $a = 1, b = 0$ and $a = 2, b = 2$ are the two "stable squares" in that if the program lands in that square, they will not move to another square because all surrounding squares are worse solutions than that one.

Let $p_{0,1}$ indicate the probability that the program will find the optimal solution if it begins at $a = 0, b = 1$. Let $p_{1,1}$ indicate the probability that the program will find the optimal solution if it begins at $a = 1, b = 1$.

We can calculate $p_{0,1}$ and $p_{1,1}$ as follows. The directions taken that correspond to each probability are listed here.

$$p_{1,1} = \underbrace{\frac{1}{4}}_{\text{right}} + \underbrace{\frac{1}{4}}_{\text{down}} + \underbrace{\frac{1}{4}}_{\text{left}} (p_{0,1})$$

$$p_{0,1} = \underbrace{\frac{1}{4}}_{\text{down}} + \underbrace{\frac{1}{4}}_{\text{right}} (p_{1,1}) + \underbrace{\frac{1}{4}}_{\text{left}} (p_{0,1})$$

We can substitute $p_{1,1}$ into $p_{0,1}$, since the former has fewer variables. We have

$$p_{0,1} = \frac{1}{4} + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{4}p_{0,1}\right) + \frac{1}{4}p_{0,1}$$

This can be simplified as follows:

$$\begin{aligned} p_{0,1} &= \frac{1}{4} + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{4}p_{0,1}\right) + \frac{1}{4}p_{0,1} \\ &= \frac{1}{4} + \frac{1}{8} + \frac{1}{16}p0,1 + \frac{1}{4}p_{0,1} \\ &= \frac{3}{8} + \frac{5}{6}p_{0,1} \end{aligned}$$

This is an especially clean equation. A computer program was used to set an initial value t (this is arbitrary), and the function `t -> 3/8 + (5/6)*t` was iterated upon it several thousand times until it converged to $0.54545454\ldots$. This is $\frac{6}{11}$. Thus, $p_{1,1} = \frac{1}{2} + \frac{1}{4} \cdot \frac{6}{11} = \frac{7}{11} = 0.6363\ldots$. This differs from the solution in the answer key, which is $\frac{2}{3}$, but it's very close.

Moreover, using another method assuming slightly different rules yields the same solution.

If the program initializes and chooses to go right or down, it will find the correct solution. If it chooses to go left, there's a $\frac{1}{3}$ chance it will go down, which will lead it to the correct answer. So far, we have a $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{3} = \frac{7}{12}$ probability that the program will find the correct solution. However, if the program goes left after its initialization, there is also a $\frac{1}{3}$ chance that the program will end up back at $a = 0, b = 0$ if it goes right (assuming walls are not counted as viable solutions). Thus, there is a $\frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$ chance whenever the program is at $a = 0, b = 0$ that they will end up back there again, and the game can be said to "restart".

The probability the program will end up at $a = 2, b = 2$ is thus

$$\frac{7}{12} + \frac{1}{12} \cdot \frac{7}{12} + \left(\frac{1}{12}\right)^2 \frac{7}{12} + \left(\frac{1}{12}\right)^3 \frac{7}{12} + \ldots$$

We can rewrite this as

$$a = \frac{7}{12^1} + \frac{7}{12^2} + \frac{7}{12^3} + \frac{7}{12^4} + \ldots$$

We have that

$$12a = 7 + \frac{7}{12^1} + \frac{7}{12^2} + \frac{7}{12^3} + \frac{7}{12^4} + \ldots$$

Subtracting yields

$$12a - a = 7 \to a = \frac{7}{11}$$

Part B Problem: Suppose now that the program has an error chance of 10%: 10% of the time, the program will switch to a worse candidate, or will refrain from switching to a better one. How will this affect the outcome? Be qualitative but concrete; we don't really have the tools to be quantitative here.

Part B Solution: This will eliminate the possibility of permanent "stable squares". When the program runs a sufficient number of times, it will have been able to traverse every single square in the board. Thus, even if it moves upwards directly after initialization at $a = 1, b = 1$ or takes another path towards $a = 1, b = 0$, it will not reside there forever and will eventually break away. This comes at the cost of the program not staying at at the optimal solution forever. One could build into the program a "stopping indicator" in which the programs stops after it's traversed all the squares, but in that case a naive grid search would fare better in terms of efficiency.