Homework 6

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Problem 18.4

Question: These graphs represent periodic functions. Describe the period in each case.

Solution: From left to right, top to bottom:

- The period is the "diameter" of one semicircle.
- The period is the horizontal length one "dumbbell"/line segment occupies.
- The period is the horizontal length from initial medium zag down to big zag up to biggest zag down to big zag up.
- The period is the "diameter" of one semicircle multiplied by 4.
- The period is the horizontal length of one line segment multiplied by 2.

Problem 19.7

Context: The voltage output(in volts) of an electrical circuit at time t seconds is given by the function

$$V(t) = 2^{3\sin(5\pi t - 3\pi)+1}$$

Part A Question: What is the initial voltage output of the circuit?

Part A Solution: The initial voltage occurs at time $t = 0$, yielding the expression $V(t) = 2^{3\sin(-3\pi)+1}$. Since $\sin(-3\pi) = 0$, the expression is equivalent to 2^1, which equals 1. The initial voltage output of the circuit is thus 2 volts.

Part B Question: Is the voltage output of the circuit ever equal to zero? Explain.

Part B Solution: No.

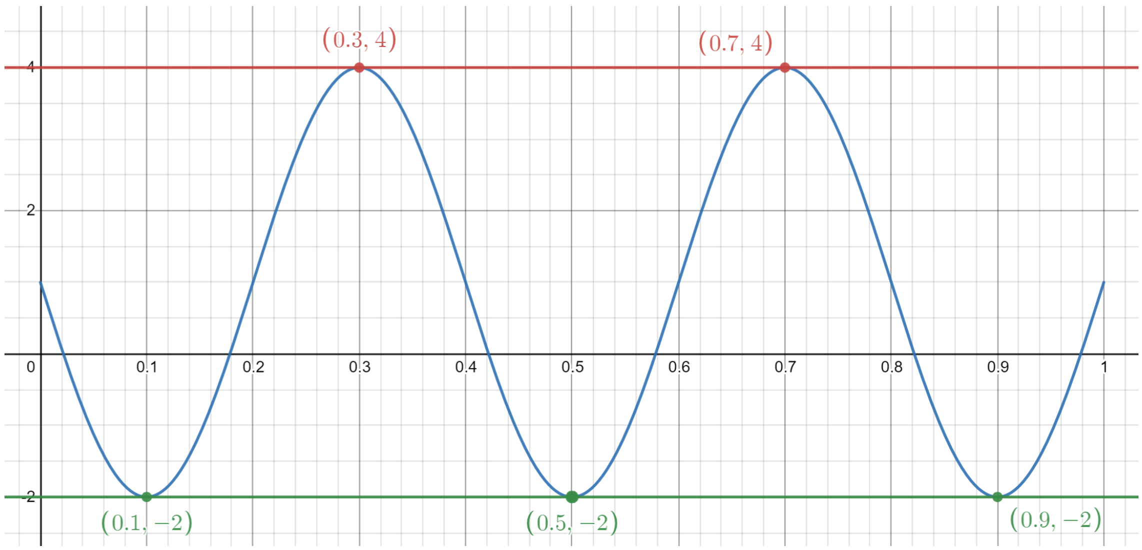
$$2^{3\sin(5\pi t - 3\pi)+1} = 0$$

$$3\sin(5\pi t - 3\pi) + 1 = \log_2(0)$$

$\log_2(0)$ is undefined, and therefore the equation cannot be solved.

Part C Question: The function $V(t) = 2^{p(t)}$, where $p(t) = 3\sin(5\pi t - 3\pi) + 1$. Put the sinusoidal function $p(t)$ in standard form and sketch the graph for $0 \le t \le 1$. Label the coordinates of the extrema on the graph.

Part C Solution: $3\sin(5\pi t - 3\pi) + 1$ can be written in standard form as $3\sin\left(\frac{2\pi}{\frac{2}{5}}\left(t - \frac{3}{5}\right)\right) + 1$. The minima and maxima are located at $1 - 3$ and $1 + 3$, yielding -2 and 4, respectively. The period is $\frac{2\pi}{5}$, meaning that one maximum of the unshifted sinusoidal is located at $x = \frac{\frac{2}{5}}{4} = 0.1$. After a shift of three-fifths right, this maximum is located now at 0.7. All maxima can be found by adding or abstracting multiples of the period, yielding $0.7, 0.3$, etc. The minima can be found in-between adjacent maxima.



Part D Question: Calculate the maximum and minimum voltage output of the circuit.

Part D Solution: The maximum value of an "untouched" sine function is 1. When the amplitude is multiplied by three and the graph is shifted upwards 1, the maximum value becomes $1 \cdot 3 + 1 = 4$ volts. Likewise, the minimum value is $-1 \cdot 3 + 1 = -2$ volts. Exponentiating these yields a range of $\frac{1}{2} \leq V(t) \leq 16$.

Part E Question: During the first second, determine when the voltage output of the circuit is 10 volts.

Part E Solution:

$$\begin{aligned}
2^{3\sin(5\pi t - 3\pi)+1} &= 10 \\
3\sin(5\pi t - 3\pi) + 1 &= \log_2(10) \\
\sin(5\pi t - 3\pi) &= \frac{\log_2(10) - 1}{3} \\
5\pi t - 3\pi &= \arcsin\left(\frac{\log_2(10)-1}{3}\right) + 2\pi n \text{ or } \pi - \arcsin\left(\frac{\log_2(10)-1}{3}\right) + 2\pi n \\
t &= \frac{\arcsin\left(\frac{\log_2(10)-1}{3}\right) + 2\pi n + 3\pi}{5\pi} \text{ or } \frac{\pi - \arcsin\left(\frac{\log_2(10)-1}{3}\right) + 2\pi n + 3\pi}{5\pi}
\end{aligned}$$

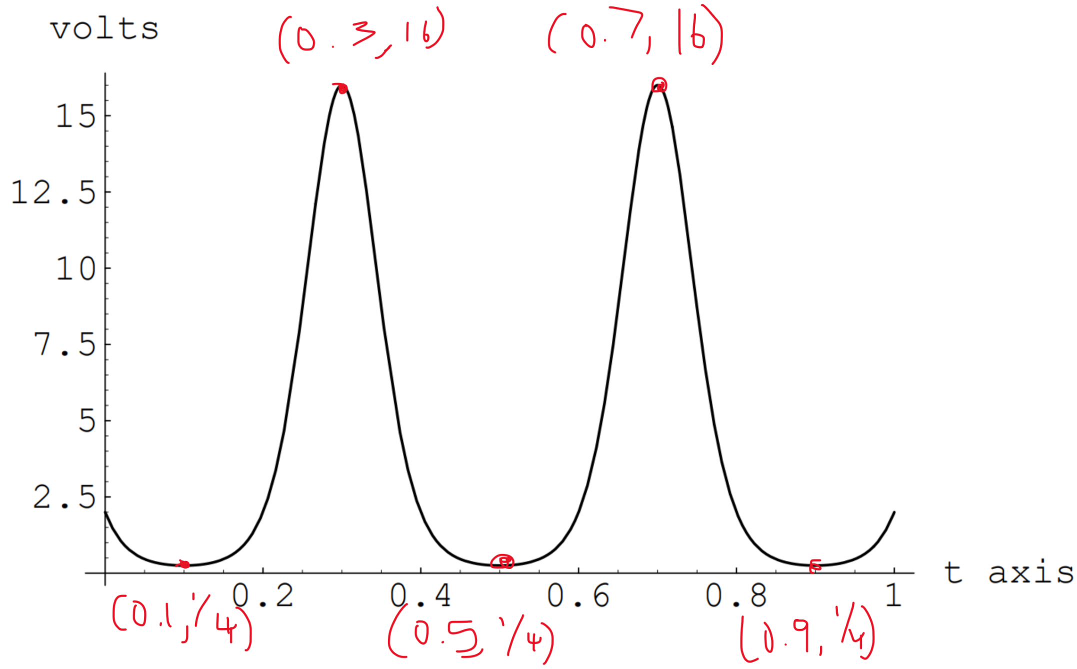
We have the following solutions by using different values of n for times t at which the voltage output is 10 volts with the constraint $0 \leq t \leq 1$:

- $\frac{\arcsin\left(\frac{\log_2(10)-1}{3}\right)+2\pi(-1)+3\pi}{5\pi} \approx 0.25634.$
- $\frac{\arcsin\left(\frac{\log_2(10)-1}{3}\right)+2\pi(0)+3\pi}{5\pi} \approx 0.65634.$
- $\frac{\pi-\arcsin\left(\frac{\log_2(10)-1}{3}\right)+2\pi(-1)+3\pi}{5\pi} \approx 0.34365.$
- $\frac{\pi-\arcsin\left(\frac{\log_2(10)-1}{3}\right)+2\pi(0)+3\pi}{5\pi} \approx 0.74365.$

Thus, the four solutions are at 0.27, 0.34, 0.66, and 0.74 seconds.

Part F Problem: A picture of the graph of $y = V(t)$ on the domain $0 \leq t \leq 1$ is given; label the coordinates of the extrema on the graph.

Part F Solution: The maxima and minima of $3\sin\left(\frac{2\pi}{\frac{2}{5}}\left(x - \frac{3}{5}\right)\right) + 1$ were found in part C. Since 2^x is a monotonically increasing function, the maxima and minima of $f(x)$ will also be the maxima and minima of $2^{f(x)}$. Thus, the maxima of $V(t)$ is $2^4 = 16$, and the minima is $2^{-2} = \frac{1}{4}$. These are located at $0.3 + 0.2k$ and $0.1 + 0.2k$, respectively.



Part G Problem: Restrict the function $V(t)$ to the domain $0.1 \leq t \leq 0.3$; explain why this function has an inverse and find the formula for the inverse rule. Restrict the function $V(t)$ to the domain $0.3 \leq t \leq 0.5$; explain why this function has an inverse and find the formula for the inverse rule.

Part G Solution: The segment of $V(t)$ from $0.1 \leq t \leq 0.3$ is one-to-one, and thus there exists an inverse function for it. Likewise, the segment of $V(t)$ from $0.3 \leq t \leq 0.5$ is one-to-one, and there exists an inverse function for it.

$$2^{3\sin(5\pi y - 3\pi)+1} = t$$
$$3\sin(5\pi y - 3\pi) + 1 = \log_2(t)$$
$$\sin(5\pi y - 3\pi) = \frac{\log_2(t) - 1}{3}$$
$$5\pi y - 3\pi = \arcsin\left(\frac{\log_2(t) - 1}{3}\right) + 2\pi n \text{ or } \pi - \arcsin\left(\frac{\log_2(t) - 1}{3}\right) + 2\pi n$$
$$y = \frac{\arcsin\left(\frac{\log_2(x)-1}{3}\right) + 2\pi n + 3\pi}{5\pi} \text{ or } \frac{\pi - \arcsin\left(\frac{\log_2(x)-1}{3}\right) + 2\pi n + 3\pi}{5\pi}$$

We can test which solution is returned by setting x to some value between the minimum and maximum y-values of the sinusoidal, like 5. Using $n = 0$, $\frac{\arcsin\left(\frac{\log_2(5)-1}{3}\right)+2\pi n+3\pi}{5\pi} \approx 0.62904$. This suggests that the inverse function with $n = 0$ is the inverse of the sinusoidal with domain restricted to $0.5 \leq t \leq 0.7$. Subtracting 2π by setting $n = -1$ will thus return the inverse for the domain $0.1 \leq t \leq 0.3$. We can confirm this: $\frac{\arcsin\left(\frac{\log_2(5)-1}{3}\right)+2\pi(-1)+3\pi}{5\pi} \approx 0.22904$. Since the domain range $0.3 \leq t \leq 0.5$ is on the down-sloping side, we will need to use the second arcsine solution $(\pi - \arcsin(\dots) + 2\pi n)$. Since this solution is right-adjacent to the prior one, $n = -1$. Thus, $\frac{\pi-\arcsin\left(\frac{\log_2(x)-1}{3}\right)+3\pi}{5\pi}$ is the inverse of the sinusoidal from range $0.3 \leq t \leq 0.5$. We can confirm this: $\frac{\pi-\arcsin\left(\frac{\log_2(5)-1}{3}\right)+2\pi(-1)+3\pi}{5\pi} \approx 0.37095$.

Thus, the inverse of $V(t)\{0.1 \leq t \leq 0.3\}$ is $\frac{\arcsin\left(\frac{\log_2(x)-1}{3}\right)+\pi}{5\pi}$ and the inverse of $V(t)\{0.3 \leq t \leq 0.5\}$ is $\frac{-\arcsin\left(\frac{\log_2(x)-1}{3}\right)+2\pi}{5\pi}$.

Problem 20.1

Part C Problem: Find four values of x that satisfy the equation $5\sin\left(2x^2 + x + 1\right) = 2$.

Part C Solution:

$$5\sin\left(2x^2+x+1\right)=2$$
$$\sin\left(2x^2+x+1\right)=\frac{2}{5}$$
$$2x^2+x+1=\arcsin\left(\frac{2}{5}\right)+2\pi n$$

Following one possibility, in which $n=0$:

$$2x^2+x+1=\arcsin\left(\frac{2}{5}\right)$$
$$2x^2+x+1-\arcsin\left(\frac{2}{5}\right)=0$$
$$x=\frac{-1\pm\sqrt{1-4(2)\left(1-\arcsin\left(\frac{2}{5}\right)\right)}}{2(2)}$$
$$x=\frac{-1\pm\sqrt{8\arcsin\left(\frac{2}{5}\right)-7}}{4}$$

Following another possibility, in which $n=1$:

$$2x^2+x+1=\arcsin\left(\frac{2}{5}\right)+2\pi$$
$$2x^2+x+1-\arcsin\left(\frac{2}{5}\right)-2\pi=0$$
$$x=\frac{-1\pm\sqrt{1-4(2)\left(1-\arcsin\left(\frac{2}{5}\right)\right)-2\pi}}{2(2)}$$
$$x=\frac{-1\pm\sqrt{8\arcsin\left(\frac{2}{5}\right)-7-2\pi}}{4}$$

Thus, four solutions are $x=\frac{-1\pm\sqrt{8\arcsin\left(\frac{2}{5}\right)-7}}{4}$ and $x=\frac{-1\pm\sqrt{8\arcsin\left(\frac{2}{5}\right)-7-2\pi}}{4}$.

Problem 20.4

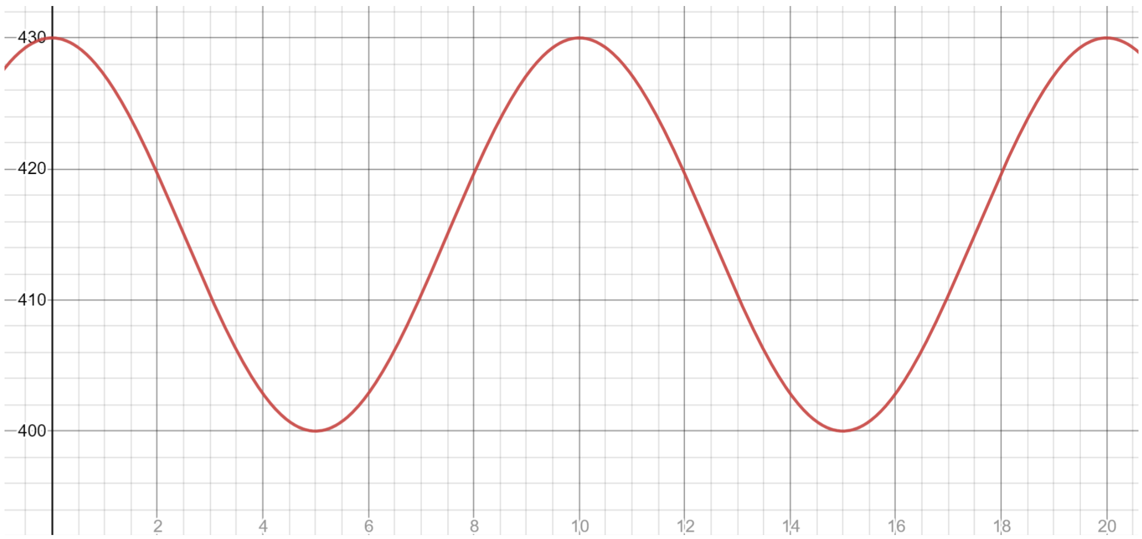
Context: Hugo bakes world famous scones. The key to his success is a special oven whose temperature varies according to a sinusoidal function; assume the temperature (in degrees Fahrenheit) of the oven t minutes after inserting the scones is given by

$$y=s(t)=15\sin\left(\frac{\pi}{5}t-\frac{3\pi}{2}\right)+415$$

Part A Problem: Find the amplitude, phase shift, period and mean for $s(t)$ then sketch the graph on the domain $0\le t\le 20$.

Part A Solution: Rewriting yields $s(t)=15\sin\left(\frac{\pi}{5}\left(t-\frac{15}{2}\right)\right)+415=y=15\sin\left(\frac{2\pi}{10}\left(t-\frac{15}{2}\right)\right)+415$. Correspondingly, we can find that:

- ...the amplitude is 15 degrees
- ...the mean is 415 degrees
- ...the period is 10 minutes
- ...the phase shift is $\frac{15}{2}$ minutes



Part B Problem: What is the maximum temperature of the oven? Give all times when the oven achieves this maximum temperature during the first 20 minutes.

Part B Solution: The maximum temperature is the mean plus the amplitude, or 430 degrees. If the sinusoidal were not shifted, the first maximum would have occurred at $t = \frac{10}{4} = 2.5$ minutes. Because the sinusoidal has been shifted rightward $\frac{15}{2}$ minutes, the maximum is now at 10 minutes. Subtracting and adding multiples of the period yield maximums in the first 20 minutes at $t = 0, 10, 20$.

Part C Problem: What is the minimum temperature of the oven? Give all times when the oven achieves this minimum temperature during the first 20 minutes.

Part C Solution: The minimum temperature can be found by subtracting the amplitude from the mean, which yields $415 - 15 = 400°\text{F}$ The minimum temperatures are located in between adjacent minimum temperatures $x - axis$-wise. Once a solution for minimum temperatures is found, the others can be found by adding or subracting multiples of the period. Hence, the minimum temperatures occur during the first 20 minutes at $t = 5, 10$.

Part D Problem: During the first 20 minutes of baking, calculate the total amount of time the oven temperature is at least 410°F.

Part D Solution: We can find the time $0 < t < 5$ at which $s(t) = 410$ and quadruple the quantity to arrive at the answer. We can do this because, conveniently, there exists two "humped" areas within the domain $0 \leq t \leq 20$ (albeit one is split in half across the two ends).

$$\begin{aligned}
15\sin\left(\frac{2\pi}{10}\left(t - \frac{15}{2}\right)\right) + 415 = 410 \\
\sin\left(\frac{2\pi}{10}\left(t - \frac{15}{2}\right)\right) = -\frac{1}{3} \\
\frac{2\pi}{10}\left(t - \frac{15}{2}\right) = \pi - \arcsin\left(-\frac{1}{3}\right) + 2\pi n \\
t = \frac{5\left(\pi - \arcsin\left(-\frac{1}{3}\right) + 2\pi n\right)}{\pi} + \frac{15}{2}
\end{aligned}$$

Note that we use $\pi - \arcsin(\dots)$ rather than $\arcsin(\dots)$ because the solution in the domain we want falls on the left side of a circle (when visualization the sine curve as movement around it). Using $n = -1$ yields a result that satisfies our domain constraint, and we have that $t = 3.04086$. Multiplying by 4 yields a total time when the oven temperature is at least 410 degrees Fahrenheit of ≈ 12.16346 minutes.

Part E Problem: During the first 20 minutes of baking, calculate the total amount of time the oven temperature is at most 425°F.

Part E Solution: Like in Part D, we can find the time $0 \leq t \leq 5$ where $s(t) = 425$. However, the solution will be $20 - 4t$, since the problem asks for the time in which the temperature is less than or equal to 425 degrees. The logic for multiplying the time by 4 is the same as in Part D; so is the logic for using $\pi - \arcsin(\dots)$ rather than $\arcsin(\dots)$.

$$\begin{aligned}
15\sin\left(\frac{2\pi}{10}\left(t - \frac{15}{2}\right)\right) + 415 &= 425 \\
\sin\left(\frac{2\pi}{10}\left(t - \frac{15}{2}\right)\right) &= \frac{2}{3} \\
\frac{2\pi}{10}\left(t - \frac{15}{2}\right) &= \pi - \arcsin\left(\frac{2}{3}\right) + 2\pi n \\
t &= \frac{5\left(\pi - \arcsin\left(\frac{2}{3}\right) + 2\pi n\right)}{\pi} + \frac{15}{2}
\end{aligned}$$

Using $n = -1$ yields $t \approx 1.33860$ minutes. Substituting into $20 - 4t$ yields a total time the oven temperature is at most 425 degrees Fahrenheit of 14.64559 minutes.

Part F Problem: During the first 20 minutes of baking, calculate the total amount of time the oven temperature is between 410 degrees and 425 degrees.

Part F Solution: We found in Parts D and E that

$$s(3.04086) = 410$$

$$s(1.33860) = 425$$

The time in which the temperature is between 410 degrees and 425 degrees is $3.04086 - 1.33860 = 1.70226$. Multiplying this by 4 (a visual inspection of the graph demonstrates that this curved portion occurs four times from $0 \leq t \leq 20$) yields 6.80904 minutes.

Problem 20.5

Problem: The temperature in Gavin's oven is a sinusoidal function of time. Gavin sets his oven so that it has a maximum temperature of 300 degrees F and a minimum temperature of 240 degrees. Once the temperature hits 300 degrees, it takes 20 minutes before it is 300 degrees again. Gavin's cake needs to be in the oven for 30 minutes at temperatures at or above 280 degrees. He puts the cake into the oven when it is at 270 degrees and rising. How long will Gavin need to leave the cake in the oven?

Solution: We can write a sinusoidal for Gavin's oven (phase shift is arbitrary):

$$o(x) = 30\sin\left(\frac{2\pi}{20}x\right) + 270$$

Gavin puts in the cake when it is at 270 degrees and rising. This, conveniently, is the mean, which means that the value of x at which $o(x) = 270$ is the average of the values of x such that $o(x) = 240$ and $o(x) = 300$ (the minimum and maximum outputs of $o(x)$), given that they are adjacent. Reasoning this out from the shape of the constructed sinusoidal, one minimum occurs at $x_{\min} = 0 - \frac{20}{4} = -5$ and the other at $x_{\max} = 0 + \frac{20}{4} = 5$. These two are adjacent and $x_{\min} < x_{\max}$ (the temperature is "on the rise" at values in between $x_{\min}$ and $x_{\max}$). Thus, at value $x = 0$, $o(x) = 270$.

Next, we must find the time at which Gavin's oven reaches 280 degrees.

$$\begin{aligned}
30\sin\left(\frac{2\pi}{20}x\right) + 270 &= 280 \\
\sin\left(\frac{\pi}{10}x\right) &= \frac{1}{3} \\
\frac{\pi}{10}x &= \arcsin\left(\frac{1}{3}\right) + 2\pi n \\
x &= \frac{10\left(\arcsin\left(\frac{1}{3}\right) + 2\pi n\right)}{\pi}
\end{aligned}$$

Note that we used $\arcsin\left(\frac{1}{3}\right)$ and not $\pi - \arcsin\left(\frac{1}{3}\right)$ because we are concerned with solutions at which the temperature is rising. Using $n = 0$ yields $x \approx 1.08173$, which satisfies the condition $0 < x < 5$ (the oven should reach 280 degrees after it reaches 270 degrees, but before it reaches 300 degrees).

We can find the interval of time at which Gavin's oven, within the length of one period, is at or over 280 degrees. We know that the oven's temperature is over 270 degrees for $\frac{1}{2}$ the period, or 10 minutes. Moreover, this interval of 10 minutes is positioned "symmetrically" around the symmetric "bump" of the sinusoidal. Within this 10-minute interval, at ≈ 1.08173 minutes the oven rose above 280 degrees. This means that at $\approx 10 - 1.08173$ minutes the oven will fall back below 280 degrees. Thus, within one period, the oven's temperature is at or above 280 degrees for $\approx 10 - 2(1.08173) = 7.83654$ minutes.

Gavin needs his cake to be in the oven for at least 30 minutes, which means his cake should be in the oven for at least $\text{floor}\left(\frac{30}{7.83654}\right) = 3$ cycles. A complete cycle takes 20 minutes, meaning that the cake is in the oven for at least $20 \cdot 3 = 60$ minutes. After he completes 3 cycles, the cake has been baking at a temperature of 280 degrees or higher for $3 \cdot 7.83654 = 23.50962$ minutes, meaning it needs to bake for $30 - 23.50962 = 6.49038$ more minutes at 280 degrees or higher after 3 cycles. However, we need to add 1.08173 minutes to account for the discrepancy between the temperature of the end of the 3 cycles (at 270 degrees) and the temperature at which Gavin's cake needs 6.49038 more minutes (at 280 degrees). In total, Gavin's cake takes $60 + 1.08173 + 6.49038 =$ 67.57 minutes in the oven.

Additional Problem

Problem: In the spreadsheet, the high temperatures in Seattle are recorded for every six hours during the week 5/1-5/7. Find (a) a best-fit sinusoidal under the assumption that the period of the temperature is 1 and that 12:00 AM on 5/1 was a minimum; and (b) a best-fit sinusoidal under the assumption that the mean is correct and that the amplitude is the furthest the data gets from the mean. (c) Compare the accuracy of the two models.

Solution: First, the dates were transformed such that 5/1 12:00 AM corresponded with 0, 5/1 6:00 AM corresponded with $\frac{1}{4}$, 5/1 12:00 PM corresponded with $\frac{1}{2}$, 5/1 6:00 PM corresponded with $\frac{3}{4}$, 5/2 12:00 AM corresponded with 1, and so on.

Assuming that the period of the temperature is 1 (presumably, 1 day) and that 12:00 AM on 5/1 is the minimum, we can use the following sinusoidal:

$$A \sin\left(\frac{2\pi}{1}\left(x - \frac{1}{4}\right)\right) + D$$

The phase shift right of one-fourth the period positions the sinusoidal such that at time "0" the model returns the minimum. The best-fit sinusoidal, after transforming the independent variable with $t(x) = 2\pi\left(x - \frac{1}{4}\right)$, yields $A \approx 4.714285714$ and $D \approx 56.32142857$. The best-fit sinusoidal under these assumptions is thus

$$4.714 \sin\left(\frac{2\pi}{1}\left(x - \frac{1}{4}\right)\right) + 56.321$$

.

Assuming that the mean is correct and the amplitude is furthest the data gets from the mean, we have $D = \bar{y} \approx 56.3214$ and $A = \max\left(|y - \bar{y}|\right) \approx 10.3214$. We can use the following sinusoidal framework:

$$10.3214 \sin\left(B\left(x - C\right)\right) + 56.3214$$

The transformation $t(x) = \arcsin\left(\frac{x - 56.3214}{10.3214}\right)$ was first performed. Then, the following transformation was applied with the following procedure:

1. Locate the current row number n, beginning at zero, and the row value v.
2. Calculate $n_p = \text{floor}\left(\frac{n}{2}\right)$.
3. If $\mod(n, 2) = 0$, return $v + n_p \cdot 2\pi$.
4. If $\mod(n, 2) = 1$, return $\pi - v + n_p \cdot 2\pi$.

The resulting line of best fit calculation yielded a slope of 12.5961 and a y-intercept of -0.11223. Thus, the best-fit sinusoidal under this set of assumptions is

$$10.3214 \sin\left(12.5961\left(x - 0.11223\right)\right) + 56.3214$$

The RMSE of the first model is 3.8124, whereas the latter's is 10.8826. The first model is much better than the latter. When inspecting the data and model predictions visually, it is clear that setting the amplitude as the furthest the data gets from the mean in the second model yields an amplitude much too wide to handle most of the data. The assumptions in the first model on the phase shift and the period are prone, of course, to some error, but were reasonable enough and not quite as significant in terms of RMSE to model performance than overshooting the amplitude.