Homework 5

Andre Ye

10 May 2021

Problem 15.6

Context: An aircraft is flying at the speed of 500 mph at an elevation of 10 miles above the earth, beginning at the North pole and heading South along the Greenwich meridian. A spy satellite is orbiting the earth at an elevation of 4800 miles above the earth in a circular orbit in the same plane as the Greenwich meridian. Miraculously, the plane and satellite always lie on the same radial line from the center of the earth. Assume the radius of the earth is 3960 miles.

Part A Problem: When is the plane directly over a location with latitude $74°30'18''$ N for the first time?

Part A Solution: This location requires traveling an angle of $90 - \left(74 + \frac{30}{60} + \frac{18}{3600}\right) = 15.495$ degrees. This is a distance of circumference of orbit $\cdot \frac{15.495}{360} = 7940\pi \cdot \frac{15.495}{360} \approx 1073.641907$ miles. Since the aircraft flies at 500mph, it will take approximately $\frac{1073.641907}{500} \approx 2.15$ hours.

Part B Problem: How fast is the satellite moving?

Part B Solution: The aircraft moves at 500 mile per hour at 10 miles above Earth. This means that it travels along a circle with radius $3960 + 10 = 3970$ miles. The total circumference of this circle is $2 \cdot 3970 \cdot \pi = 7940\pi$. Every hour, the aircraft travels $\frac{500}{7940\pi}$th the distance of one complete rotation. If the satellite can always "keep up" with the aircraft in that both are located on the same radial line from the center of the Earth, it must travel the same fraction of one rotation as the aircraft does. The spy satellite is flying at 4800 miles above Earth, meaning that it travels along a circle with radius $3960 + 4800 = 8760$ miles. If it travels at some speed s in mile per hour, each hour it will travel $\frac{s}{17520\pi}$th of a complete rotation. Setting this equal to the derived fraction-of-full-rotation-per-hour value for the aircraft allows us to find s.

$$\begin{aligned}\frac{s}{17520\pi} &= \frac{500}{7940\pi} \\ s &= \frac{500 \cdot 17520\pi}{7940\pi} \\ &= \frac{500 \cdot 17520}{7940} \\ &\approx 1103.27 \text{ miles per hour}\end{aligned}$$

Part C Problem: When is the plane directly over the equator and how far has it traveled?

Part C Solution: To reach the equator in a direct path from the North pole, the plane will need to travel 90 degrees. This is a distance of $7940\pi \times \frac{90}{360} = 1985\pi \approx 6236.06$ miles. Since the plane travels at 500 miles per hour, it will take the plane $\frac{6236.06}{500} \approx 12.47$ hours to reach the equator.

Part D Problem: How far has the satellite traveled when the plane is directly over the equator?

Part D Solution: When the plane is over the equator, the satellite is also over the equator, meaning that it has traveled 90 degrees. This distance can be found as $17520\pi \times \frac{90}{360} = 4380\pi \approx 13760.17$ miles.

Problem 15.8

Problem: Matilda is planning a walk around the perimeter of Wedge Park, which is shaped like a circular wedge, as shown below. The walk around the park is 2.1 miles, and the park has an area of 0.25 square miles. If θ is less than 90 degrees, what is the value of the radius, r?

Solution: We have the following information, corresponding to what the problem tells us about the perimeter and the area:

$$r + r + \frac{2\pi r}{\frac{\theta}{360^\circ}} = 2.1$$
$$\frac{\pi r^2}{\frac{\theta}{360^\circ}} = 0.25$$
$$\theta < 90^\circ$$

We can simplify these as such:

$$r + r + \frac{2\pi r}{\frac{\theta}{360^\circ}} = 2.1 \implies 2r + \frac{720^\circ \pi r}{\theta} = 2.1 \implies 2r\theta + 720^\circ \pi r = 2.1\theta \implies 720^\circ \pi r = \theta\left(2.1 - 2r\right)$$
$$\implies \theta = \frac{720^\circ \pi r}{(2.1 - 2r)}$$

$$\frac{\pi r^2}{\frac{\theta}{360}} = 0.25 \implies 360\pi r^2 = 0.25\theta \implies \theta = \frac{360\pi r^2}{0.25}$$

Setting these equal to each other:

$$\begin{aligned}
\frac{720^\circ \pi r}{(2.1 - 2r)} &= \frac{360^\circ \pi r^2}{0.25} \\
\frac{0.25 \cdot 720}{(2.1 - 2r)} &= 360r \\
180 &= 360r\left(2.1 - 2r\right) \\
0 &= 2r\left(2.1 - 2r\right) - 1 \\
0 &= -4r^2 + 4.2r - 1 \\
40r^2 + 42r - 10 &= 0 \\
r &= \frac{-42 \pm \sqrt{42^2 - 4\left(-40\right)\left(-10\right)}}{2\left(-40\right)} \\
&= \frac{-42 \pm 2\sqrt{41}}{2\left(-40\right)} \\
&= \frac{21 \pm \sqrt{41}}{40}
\end{aligned}$$

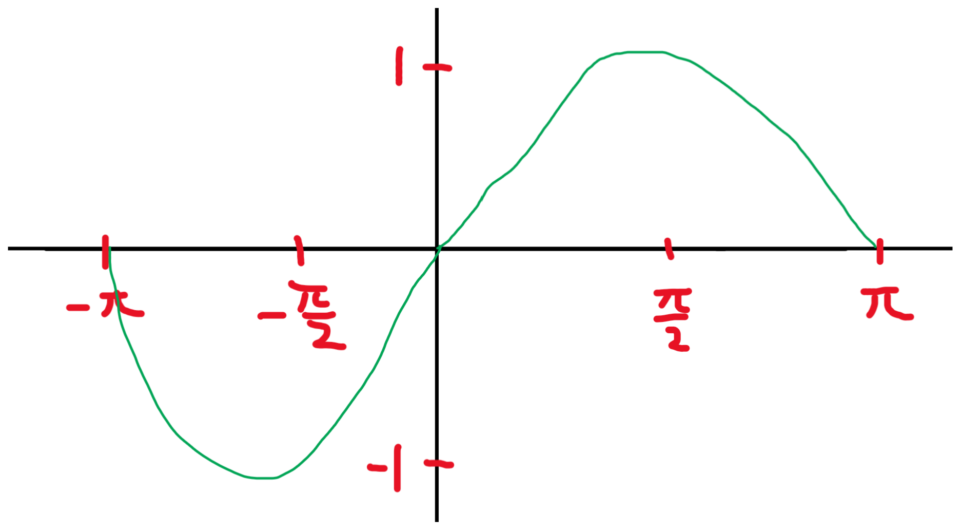
The two solutions for r end up being $\frac{21-\sqrt{41}}{40}$ and $\frac{21+\sqrt{41}}{40}$. One is larger and the other is smaller, there is only one correct answer, and θ must be less than 90 degrees. If r becomes larger, in order for the perimeter and area to remain the same, θ must become smaller. If the smaller value of r is the correct answer, then the larger value of r would also be correct, since a larger value of r yields a smaller value of θ that would satisfy the requirement $\theta < 90^\circ$ given that the smaller value of r is correct (i.e., it yielded a larger value of θ that still satisfied the requirement). Thus, only the larger value of r can be correct, since it returns one solution. The radius r is thus $\frac{21+\sqrt{41}}{40}$ miles long.

Problem 18.1

Context: Work the following problems without using ANY calculators.

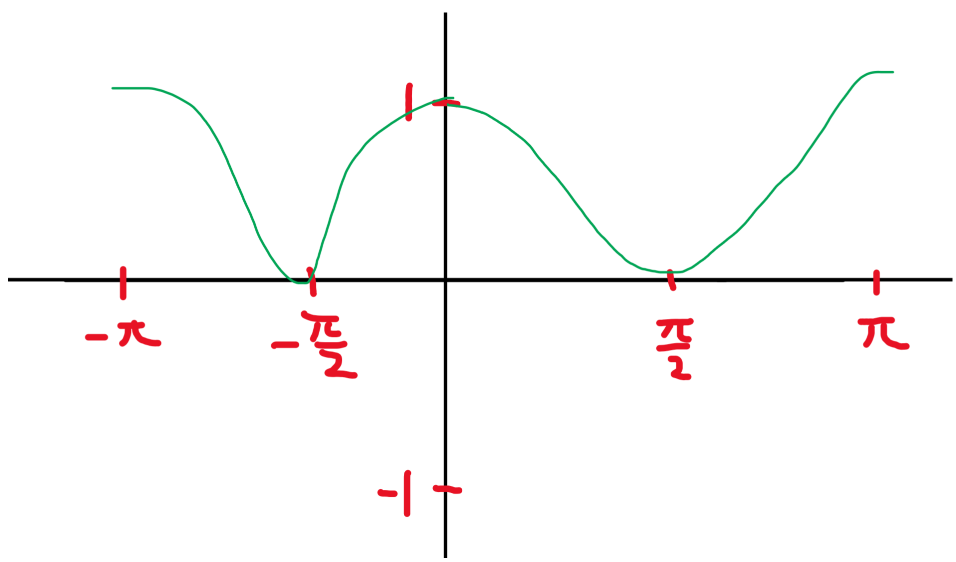
Part A Problem: Sketch $y = \sin(x)$.

Part A Solution: We have the points $(0, 0)$, $\left(\frac{\pi}{2}, 1)\right)$, $\left(-\frac{\pi}{2}, -1)\right)$, $(\pi, 0)$, $(-\pi, 0)$



Part B Problem: Sketch $y = \sin^2(x)$.

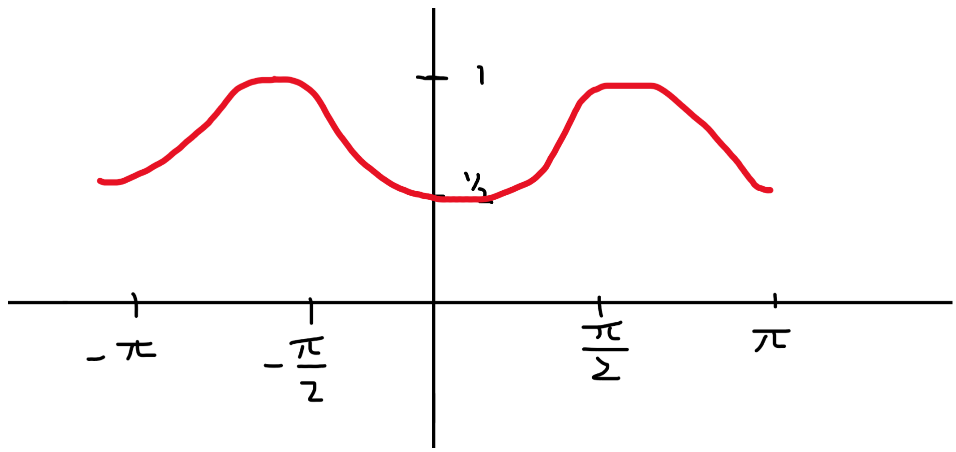
Part B Solution: Squaring values with an absolute value less than 1 yields a number with a smaller absolute value. Thus, the squared sine function will be a little bit "flatter" than that of the sine function. The range will also be larger than or equal to 0, because squaring a negative number yields a positive one.



Part C Problem: Sketch $y = \frac{1}{1+\sin^2(x)}$.

Part C Solution: $\sin^2(x)$ has been graphed in part B. Shifting it one unit up moves the minimum value to 1 and the maximum value to 2. At points where $\sin^(x) + 1$ equals 1, $\frac{1}{1+\sin^2(x)}$ equals 1; at points where $\sin^2(x)$ equals 2, $\frac{1}{1+\sin^2(x)}$ equals $\frac{1}{2}$. Thus, the maximum and minimum of $\frac{1}{1+\sin^2(x)}$ are 1 and $\frac{1}{2}$, respectively. Moreover, the points on $\sin^2(x)$ that were previously the maximum at $y = 1$ become the minimum at $y = 0.5$, and the points that were previously on the minimum at $y = 0$ become the maximum at $y = 1$. In the spirit of sketching, this is roughly approximate to reflecting $\sin^2(x)$ over the x-axis, shrinking vertically by a factor of 2, and shifting upwards by 1 unit. This yields the following sketch:

$1 + \sin^2(x)$ yields the $\sin x$ curve such that the minimum value is 0 and the maximum value is 2. At $x = -\frac{\pi}{2} \to \infty$, $\frac{1}{1+\sin^2(x)}$ will approach ∞ since $1 + \sin^2(x)$ decreases towards 0 as x approaches $-\frac{\pi}{2}$. At $x = 0$, $x = -\pi$, and $x = \pi$, $1 + \sin x$ is equal to 1; thus, $\frac{1}{1+\sin x}$ is $\frac{1}{1} = 1$. At $x = \frac{\pi}{2}$, $1 + \sin x$ is 2 and $\frac{1}{1+\sin x}$ is $\frac{1}{2}$. We can use these points to draw the following curve:



Problem 19.1

Context: Find the amplitude, period, a phase shift and the mean of the following sinusoidal functions

Part A Problem: $y = \sin(2x - \pi) + 1$

Part A Solution: Rewriting the equation in the form $A\sin(B(x - C)) + D$ yields $1\sin\left(2\left(x - \frac{\pi}{2}\right)\right) + 1$. Using the values of these coefficients, the amplitude is 1, the phase shift is $\frac{\pi}{2}$, and the mean is 1. The period is $\frac{2\pi}{p} = 2 \implies p = \pi$. Therefore, the period is π.

Part B Problem: $y = 6\sin(\pi x) - 1$

Part B Solution: Rewriting the equation in the form $A\sin(B(x - C)) + D$ yields $6\sin(\pi(x)) - 1$. Using the values of these coefficients, the amplitude is 6, the phase shift is 0, and the mean is -1. The period is $\frac{2\pi}{p} = \pi \implies p = 2$. Therefore, the period is 2.

Part C Problem: $y = 3\sin(x + 2.7) + 5.2$

Part C Solution: Rewriting the equation in the form $A\sin(B(x - C)) + D$ yields $3\sin(1(x + 2.7)) + 5.2$. Using the values of these coefficients, the amplitude is 3, the phase shift is -2.7, and the mean is 5.2. The period is $\frac{2\pi}{p} = 1 \implies p = 2\pi$. Therefore, the period is 2π.

Part D Problem: $y = 5.6\left(\sin\left(\frac{2}{3}x - 7\right) - 12.1\right)$

Part D Solution: Rewriting the equation in the form $A\sin(B(x - C)) + D$ yields $5.6\sin\left(\frac{2}{3}\left(x - \frac{14}{3}\right)\right) - 67.76$. Using the values of these coefficients, the amplitude is 5.6, the phase shift is $\frac{14}{3}$, and the mean is -67.76. The period is $\frac{2\pi}{p} = \frac{2}{3} \implies p = \frac{2\pi}{\frac{2}{3}} = 3\pi$. Therefore, the period is 3π.

Part E Problem: $y = 2.1\sin\left(\frac{x}{\pi} + 44.3\right) - 9.8$

Part E Solution: Rewriting the equation in the form $A\sin(B(x - C)) + D$ yields $2.1\sin\left(\frac{1}{\pi}(x + 44.3\pi)\right) - 9.8$. Using the values of these coefficients, the amplitude is 2.1, the phase shift is -44.3π, and the mean is -9.8. The period is $\frac{2\pi}{p} = \frac{1}{\pi} \implies p = \frac{2\pi}{\frac{1}{\pi}} = 2\pi^2$. Therefore, the period is $2\pi^2$.

Part F Problem: $y = 3.9(\sin(22.34(x + 18)) - 11)$

Part F Solution: Rewriting the equation in the form $A\sin(B(x - C)) + D$ yields $3.9\sin(22.34(x + 18)) - 42.9$. Using the values of these coefficients, the amplitude is 3.9, the phase shift is -18, and the mean is -42.9. The period is $\frac{2\pi}{p} = 22.34 \implies p = \frac{2\pi}{22.34} = \frac{\pi}{11.17}$. Therefore, the period is $\frac{\pi}{11.17}$.

Part G Problem: $y = 11.2\sin\left(\frac{5}{\pi}(x - 9.2)\right) + 8.3$

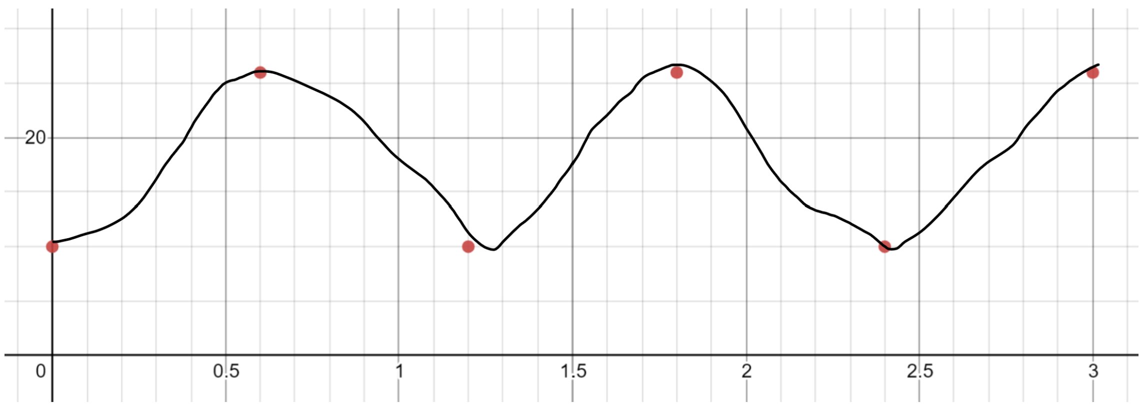
Part G Solution: The equation is already written in $A\sin(B(x - C)) + D$ form. Using the values of these coefficients, the amplitude is 11.2, the phase shift is 9.2, and the mean is 8.3. The period is $\frac{2\pi}{p} = \frac{5}{\pi} \implies p = \frac{2\pi}{\frac{5}{\pi}} = \frac{2\pi^2}{5}$. Therefore, the period is $\frac{2\pi^2}{5}$.

Problem 19.2

Context: A weight is attached to a spring suspended from a beam. At time $t = 0$, it is pulled down to a point 10 cm above the ground and released. After that, it bounces up and down between its minimum height of 10 cm and a maximum height of 26 cm, and its height $h(t)$ is a sinusoidal function of time t. It first reaches a maximum height 0.6 seconds after starting.

Part A Problem: Follow the procedure outlined in this section to sketch a rough graph of $h(t)$. Draw at least two complete cycles of the oscillation, indicating where the maxima and minima occur.

Part A Solution: The maxima is located at $y = 26$, and the minima is located at $y = 10$.



Part B Problem: What are the mean, amplitude, phase shift and period for this function?

Part B Solution: The sinusoidal function passes through the point $(0, 10)$. It hits the maximum height of 26 centimeters 0.6 seconds after starting, meaning the function also passes through the point $(0.6, 26)$ and $(1.2, 10)$.

- The mean of the minimum and maximum values is $\frac{10+26}{2} = 18$ feet.
- The amplitude is half of the range between the minimum and maximum values, which is $\frac{26-10}{2} = 8$ feet.
- The period is 1.2 hours, since the fluctuation repeats itself every $0.6 \times 2 = 1.2$ seconds.
- The phase shift is half of the shift of 0.6, which means that $C = \frac{0.6}{2} = 0.3$ hours.

Part C Problem: Give four different possible values for the phase shift.

Part C Solution: The period is 1.2. Thus, the phase shift can be $0.3 + 1.2k$, where k is some integer. This means four possible values for the phase shift are $0.3 + 1.2$, $0.3 + 2.4$, $0.3 + 3.6$, and $0.3 + 4.8$.

Part D Problem: Write down a formula for the function $h(t)$ in standard sinusoidal form; i.e. as in 19.1.1 on Page 254.

Part D Solution:

$$h(x) = A\sin(B(x - C)) + D = 8\sin\left(\frac{\pi}{0.6}(x - 0.3)\right) + 18$$

Problem 19.4

Context: Suppose the high tide in Seattle occurs at 1:00 a.m. and 1:00 p.m. at which time the water is 10 feet above the height of low tide. Low tides occur 6 hours after high tides. Suppose there are two high tides and two low tides every day and the height of the tide varies sinusoidally.

Part A Problem: Find a formula for the function $y = h(t)$ that computes the height of the tide above low tide at time t. (In other words, $y = 0$ corresponds to low tide.)

Part A Solution: It takes 12 hours for the tide to complete a "cycle"; thus, the period is 12. The amplitude is 5, since the tide deviates by 5 feet from the mean of 5 feet. The high tide occurs at 1 a.m.; we can shift an un-shifted sinusoidal function two units to the left such that the peak is located at $t = 1$. We can thus write the formula as $h(t) = 5\sin(\frac{2\pi}{12}(t+2)) + 5$, where t represents the number of hours after midnight.

Part B Problem: What is the tide height at 11:00 a.m.?

Part B Solution: $h(11) = 5\sin(\frac{2\pi}{12}(11+2)) + 5 \approx$ 7.5 feet