Homework 4

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Problem 1

Context: In (a) through (d), a two-player game is given in our standard format. Identify (i) any dominant strategies for either player, and (ii) any Nash equilibria in each game.

Part A Problem:

	A	B
A	(5,5)	(7,2)
B	(3,6)	(4,4)

Part A Solution: Player I's dominant strategy is strategy B; regardless of what Player II picks, player I is better off if they choose strategy B. Similarly, Player II's dominant strategy is strategy B. When a game has dominant strategies, the Nash equilibrium is at the intersection of the dominant strategies. Thus, the Nash equilibrium is at (B,B).

Player I Dominant Strategy	Strategy B
Player II Dominant Strategy	Strategy B
Nash Equilibria/um	(B,B)

Part B Problem:

	A	B
A	(5,0)	(0,5)
B	(0,3)	(3,0)

Part B Solution: Player I has no dominant strategy. If Player II chooses strategy A, it is rational for player I to choose strategy A; if Player II chooses strategy B, it is rational for player I to choose strategy B. Player II also has no dominant strategy. If Player I chooses strategy A, it is rational for Player II to choose strategy B; if Player I chooses strategy B, it is rational for Player II to choose strategy A. These decisions can be displayed visually by underlining which decision a player will make given the other player's decision:

	A	B
A	($\underline{5}$,0)	(0,$\underline{5}$)
B	(0,$\underline{3}$)	($\underline{3}$,0)

Because there is no one pair of strategies for which both Players would not deviate from, there is no Nash equilibria.

Player I Dominant Strategy	None
Player II Dominant Strategy	None
Nash Equilibria/um	None

Part C Problem:

	A	B
A	(5,5)	(0,0)
B	(0,0)	(1,1)

Part C Solution: Player I has no dominant strategy. If Player II chooses strategy A, it is rational for player I to choose strategy A; if Player II chooses strategy B, it is rational for player I to choose strategy B. Player II also has no dominant strategy. If Player I chooses strategy A, it is rational for Player II to choose strategy A; if Player I chooses strategy B, it is rational for Player II to choose strategy B.

These decisions can be displayed visually by underlining which decision a player will make given the other player's decision:

	A	B
A	($\underline{5}$,$\underline{5}$)	(0,0)
B	(0,0)	($\underline{1}$,$\underline{1}$)

Nash equilibria occur when both Player I and Player II choose a strategy that is advantageous for both of them given the other's choice. Thus, the two Nash equilibria in this case are the two with both reward values underlined, yielding (A,A) and (B,B).

Player I Dominant Strategy	None
Player II Dominant Strategy	None
Nash Equilibria/um	(A,A) and (B,B)

Part D Problem:

	A	B	C
A	(5,5)	(6,0)	(7,0)
B	(0,6)	(0,0)	(0,0)
C	(0,7)	(0,0)	(0,0)

Part D Solution: Player I does not have a dominant strategy; if player II chooses B or C, no strategy is better than the other to choose, whereas a dominant strategy must be better than all under strategies under any decision the other player makes. Player II, similarly, under the same logic does not have a dominant strategy.

These decisions can be displayed visually by underlining which decision a player will make given the other player's decision, when it is better than all other decisions under the same circumstances:

	A	B	C
A	(5,5)	(6,0)	($\underline{7}$,0)
B	(0,6)	(0,0)	(0,0)
C	(0,$\underline{7}$)	(0,0)	(0,0)

Nash equilibria occur when both Player I and Player II choose a strategy that is advantageous for both of them given the other's choice. There are hence no "strong" Nash equilibria. We, can, however, find six weak Nash equilibria:

	A	B	C
A	(5,5)	(6,$\underline{0}$)	($\underline{7}$,$\underline{0}$)
B	($\underline{0}$,6)	($\underline{0}$,$\underline{0}$)	($\underline{0}$,$\underline{0}$)
C	($\underline{0}$,$\underline{7}$)	($\underline{0}$,$\underline{0}$)	($\underline{0}$,$\underline{0}$)

Player I Dominant Strategy	None
Player II Dominant Strategy	None
Strong Nash Equilibria/um	None
Weak Nash Equilibria/um	(A, C), (B, B), (B, C), (C, A), (C, B), (C, C)

Problem 2

Question: Give an example of a two-player game with exactly three different Nash equilibria.

Solution: We need to have three different Nash equilibria. When a cell's tuple is composed of two rewards that are better than all other rewards in its row or column, it is a Nash equilibrium. One of the easiest ways to attain N different Nash equilibria, then, is to initiate a $N \times N$ table with the diagonals filled in with pairs of nonzero numbers and the rest of the table filled with $(0, 0)$.

	A	B	C
A	(1,1)	(0,0)	(0,0)
B	(0,0)	(2,2)	(0,0)
C	(0,0)	(0,0)	(3,3)

Problem 3

Question: The British game show Golden Balls, which aired from 2007 to 2009, included an event called Split or Steal?. In this event, two players decide how the jackpot will be distributed. Each of them is given a pair of golden balls; one of the balls has the word "Split" inside of it, the other has the word "Steal". Each player looks inside the balls to tell which is which, then chooses one (without showing it). If both players choose "Split", they each get half of the jackpot. If one chooses "Split" and the other chooses "Steal", then the one that chose "Steal" gets the whole jackpot and the other one gets nothing. But if they both choose "Steal", neither of them win anything. Present Split or Steal? as a two-player game in our standard format, and identify the dominant strategies and Nash equilibria.

Solution: Let 1 represent the jackpot. As such, $\frac{1}{2}$ represents half the jackpot. The below representation is oriented such that Player I's decisions are horizontally arranged top and Player II's decisions are vertically arranged on the left column. The first element of the tuple in each cell is Player I's reward. As such, we can represent *Golden Balls* in the following table:

	Split	Steal
Split	$(\frac{1}{2}, \frac{1}{2})$	$(1, 0)$
Steal	$(0, 1)$	$(0, 0)$

There is no dominant strategy. If a Player decides to split, it is better for the other Player to steal. However, if a Player decides to steal, both splitting and stealing are the same to the other Player (returning a reward of 0).

Finding the Nash equilibria via the "underline method":

	Split	Steal
Split	$(\frac{1}{2}, \frac{1}{2})$	$(\underline{1}, 0)$
Steal	$(0, \underline{1})$	$(0, 0)$

There are also no Nash equilibria.

Problem 4

Question: Is it possible for both players in a two-player game to play their dominant strategies, but not wind up at a Nash equilibrium? If so, give an example of a game with this property. If not, explain why not.

Solution: No, it is not possible. If a player has a dominant strategy, regardless of what the other player chooses, that strategy will yield the maximum reward out of all possible strategies that player can take. Therefore, there is never an action the other player takes that will make the player change their strategy. Nash equilibria are locations in which both players would not change their strategy if the other player changed theirs; because both players are playing their dominant strategies, by definition they play at the Nash equilibria.

Problem 5

Question: A classic example in game theory is the ultimatum game. In the ultimatum game, one player is given a sum of money; for the sake of argument, let's say this is \$6. They must then make an offer to the second player, offering to give that player any whole number of dollars between \$0 and \$6. The second player may accept or reject the offer. If they accept the offer, then each player gets their money: the second player gets the offered amount, and the first player gets the remainder of the \$6. If the second player rejects, no one gets anything. Now, as a game, it would be reasonable to represent it like so:

	Offer \$ 0	Offer \$ 1	Offer \$ 2	Offer \$ 3	Offer \$ 4	Offer \$ 5	Offer \$ 6
Accept	(6,0)	(5,1)	(4,2)	(3,3)	(2,4)	(1,5)	(0,6)
Reject	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

The dominant strategy for Player II is to accept, because accepting any offer always puts Player II in at least as good of a position as rejecting would. Given that it is the rational decision to play by the dominant strategy, Player I should expect that Player II will accept no matter what, and should therefore offer \$0. In practice, however, Player II generally rejects unless the division is roughly equal or better. Give an explanation for this phenomenon within the context of game theory. For example, you might argue that a different table would represent this game more accurately.

Solution: In this problem, one person has control over the allocation of a scarce resource – there are only six dollars – and the other person has control over whether the allocation is implemented or not. The scarcity of the problem also makes it a bit like a zero-sum game shift right three units, since each player's gain comes at the loss of the other. We can thus reframe the problem to better reflect is scarce zero-sum roots as such:

	Offer \$ 0	Offer \$ 1	Offer \$ 2	Offer \$ 3	Offer \$ 4	Offer \$ 5	Offer \$ 6
Accept	$(3, -3)$	$(2, -2)$	$(1, -1)$	$(0, 0)$	$(-1, 1)$	$(-2, 2)$	$(-3, 3)$
Reject	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

However, with this formulation we run into the problem that the frame that the "Accept" row operates in is completely different from the "Reject" one, and that comparing the two would mean that it is equally advantageous for Player II to reject a \$3 as it is for them to accept it (and equally advantageous for Player I to not have \$0 as it is for them to have it). To correct this disparity, we could set all values of rejection to $(-3, -3)$ – the lowest possible resource amount – but then it is no different from the prior situation. Fundamentally, when Player II rejects the offer, the game is no longer "scarce" in the sense that there is a non-zero quantity of resources that must be divvied among the players. We would like somehow to incorporate the meaningful "scarce"/zero-sum spirit of the reward tuples when Player II accepts, though. We can add the two tables together such that the values can be thought of as "a base level of money to account for the fact that having something is better than having nothing in the non-zero-sum-game case that Player II rejects with values that reflect the amount of money players receive relative to each other in the zero-sum-game case that Player II accepts". The result is a stretch by a factor by two around 3.

	Offer \$ 0	Offer \$ 1	Offer \$ 2	Offer \$ 3	Offer \$ 4	Offer \$ 5	Offer \$ 6
Accept	$(9, -3)$	$(7, -1)$	$(5, 1)$	$(3, 3)$	$(1, 5)$	$(-1, 7)$	$(-3, 9)$
Reject	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

There is a meaningful difference, though; Player II decides that if Player I splits the money $(5, 1)$, for instance, the value of the absolute amount of money relative to getting nothing with rejection is less strong than the actual value of that money relative to the other person accounting for scarcity. By adding together the two tables, we are essentially forcing the zero-sum world and the absolute world to work together or against each other to recommend which world Player II should enter.

With this framing, it is clear that Player II rejects \$0 and \$1 offers. In this particular framing, Player II would accept a \$2 offer, but only because we have implicitly made a choice to weight the absolute amount of money someone gets (let's call this m_a) equally to the relative amount of money someone has accounting for scarcity (let's call this m_r). We can generalize the act of adding together the tables with two parameters, α and β, such that each cell value in the new "combined" table is given by:

$$\alpha m_a + \beta m_r$$

Perhaps some people are saltier than others, so $\beta > \alpha$. Alternatively, perhaps you aren't that jealous and you value having some money more than having more money relative to the other person, so $\alpha > \beta$. If you are incredibly altruistic, $\beta < 0$; this way, any gain that you have relative to others will be penalized. With the evidence that the problem gives – that most people want an equal split or better – β would need to be slightly higher than α.

Problem 6

Question: In class, we have (or will, depending on when you're reading this) discussed the iterated Prisoner's Dilemma, in which two players play the Prisoner's Dilemma against each other repeatedly, each attempting to maximize their own total score. We saw that a limited form of retribution (choosing to defect if the opponent defected last time, but otherwise to cooperate) was generally a successful strategy.

The Volunteer Problem is a similar game for N players. Each player may choose between two actions, A and B. If any player chooses A, then each player that chose A earns a reward of 1, and each player that chose B earns a reward of 2. If no player chooses A, then no player earns any reward. Devise a "good" strategy for this game, and explain why. There is no need to find a "best" strategy; any strategy that works better on average than randomly guessing will be counted as correct, provided it is well-explained.

Solution: Let us first present the problem in table format (with Player I on top and Player I's reward as the first element of the tuple):

	A	B
A	(1,1)	(2,1)
B	(1,2)	(0,0)

There are no dominant strategies, although there are two (strong) Nash equilibria at (A,B) and (B,A). This indicates that when the two players choose different actions, they are in a position such that no one of them would find it profitable to change their action, provided the other player does not change theirs.

	A	B
A	(1,1)	($\underline{2}$,$\underline{1}$)
B	($\underline{1}$,$\underline{2}$)	(0,0)

We can outline the following characteristics of this game:

- At an cursory glance, the layout of the game is very similar to that of the Prisoner's Dilemma, which suggests that some form of retribution would have a high probability of being successful.
- A is the "safe" choice; if one player chooses action A, they are guaranteed to get 1 point, regardless of what the other player chooses.
- B is the "risky" choice; if one player chooses action B, they can either gain 2 points or 0 points, depending if the other player chooses action A or B.
- A is the "diplomatic" choice; if a player chooses action A, the other player will either gain the same amount of (or more) points as they do.
- B is the "undiplomatic" choice; if a player chooses action B, they either gain more points than the other player, or alternatively contribute to both players receiving no points.
- Being diplomatic builds trust without any absolute cost to a player, although it returns fewer rewards – although it widens the gap in points between the players, being diplomatic guarantees 1 point.
- Being undiplomatic tarnishes trust, although it gives an undiplomatic player the chance to gain more points than the other.

Using these observations, we can formulate the following strategic points:

- If one player is consistently diplomatic and the other is consistently undiplomatic, the diplomatic player is being "exploited".
- To prevent exploitation, a diplomatic player would seek to reduce the incentive for the undiplomatic player to consistently be undiplomatic by also being undiplomatic. While this comes at a short-term cost to the diplomatic player, it will likely succeed in forcing the undiplomatic player to be diplomatic again such that both have a higher gain.
- Given that consistent "undiplomacy" will lead to retribution, it is more sustainable for a player to be consistently diplomatic as opposed to being consistently undiplomatic.
- A diplomatic player should have some tolerance for "undiplomacy", since the other opponent's undiplomatic move comes at no loss for them.
- If both players decide to alternate between the Nash equilibria, they can maximize their points.

Thus, our strategy can be formulated as such: the player will alternate between being diplomatic (when the other player is undiplomatic) and undiplomatic (when the other player is diplomatic), such that you sustainably increase the number of points that you have. If the other player does not adhere to the alternating strategy and damages the player by being undiplomatic when the player is also being undiplomatic by agreement, the player will continue to be undiplomatic such that the other player is incentivized to cooperate (i.e. be diplomatic). After the other player chooses to be diplomatic and the player chooses to be undiplomatic (out of continued retribution), the alternating pattern continues.