Collingwood 9

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3.1 Context: This exercise emphasizes the "mechanical aspects" of circles and their equations.

3.1a Problem: Find an equation whose graph is a circle of radius 3 centered at (-3,4).

3.1a Solution: $\boxed{(x+3)^2 + (y-4)^2 = 9.}$

3.1b Problem: Find an equation whose graph is a circle of diameter $\frac{1}{2}$ centered at the point $(3, -\frac{11}{3})$.

3.1b Solution: The radius of such a circle is $\frac{1}{4}$ units. Thus, the equation is $\boxed{(x-3)^2 + (y+\frac{11}{3})^2 = \frac{1}{16}.}$

3.1c Problem: Find four different equations whose graphs are circles of radius 2 through (1,1).

3.1c Solution: A simple (but not too simple) method to find the four circles directly up-right, up-left, down-right, and down-left of (1,1) at 45-degree angles. If the radius of a circle of length 2 extends from (1,1) at 45-degrees, it ends at point $(1+\frac{2}{\sqrt{2}}, 1+\frac{2}{\sqrt{2}})$. In rationalized form, this is $(1+\sqrt{2}, 1+\sqrt{2})$. Similarly, three other points 2 units away from (1,1) can be constructed:

- $(1-\sqrt{2}, 1+\sqrt{2})$
- $(1+\sqrt{2}, 1-\sqrt{2})$
- $(1-\sqrt{2}, 1-\sqrt{2})$

Thus, four circle equations can be constructed:

- $(x-1-\sqrt{2})^2 + (y-1-\sqrt{2})^2 = 4$
- $(x-1-\sqrt{2})^2 + (y-1+\sqrt{2})^2 = 4$
- $(x-1+\sqrt{2})^2 + (y-1-\sqrt{2})^2 = 4$
- $(x-1+\sqrt{2})^2 + (y-1+\sqrt{2})^2 = 4$

3.1d Problem: Consider the equation $(x-1)^2 + (y+1)^2 = 4$. Which of the following points lie on the graph of this equation: (1,1), $(1, -1)$, $(1, -3)$, $(1+\sqrt{3}$,0), (0,$-1-\sqrt{3}$), (0,0).

3.1d Solution: The circle drawn by the equation $(x-1)^2 + (y+1)^2 = 4$ has a radius of $\sqrt{4} = 2$ units and is centered at (1, -1). To move this circle's location to the origin, we move each point along $\langle -1, 1 \rangle$. Therefore, the proposed points become (0,2), (0, 0), $(0, -2)$, $(\sqrt{3}, -1)$, $(-1, -\sqrt{3})$, (1,-1).

By centering the circle, all points that lie on it must be 2 units away from the origin, which is exceptionally easy to calculate. It also has the benefit of removing pesky $a + b\sqrt{c}$-type expressions, in this case converting them simply to $b\sqrt{c}$, which is much easier to square and perform operations on.

- $(0,2) \to \sqrt{0+4} = 2$
- $(0,0) \to \sqrt{0+0} = 0$

- $(0, -2) \to \sqrt{0+4} = 2$
- $(\sqrt{3}, -1) \to \sqrt{3+1} = 2$
- $(-1, -\sqrt{3}) \to \sqrt{1+3} = 2$
- $(1, -1) \to \sqrt{1+1} = \sqrt{2}$

Mapping these back to their original points via $\langle 1, -1 \rangle$, we obtain $\boxed{(1,1),\ (1,-3),\ (1+\sqrt{3},0),\ (0,-1-\sqrt{3})}$ as points on the circle.

3.2 Context: Find the center and radius of each of the following circles.

3.2a Problem: $x^2 - 6x + y^2 + 2y - 2 = 0$.

3.2a Solution: Completing the square for the x and y component, we can form the equation for a circle:

$$\begin{aligned} x^2 - 6x + y^2 + 2y - 2 &= 0 \\ x^2 - 6x + 9 + y^2 + 2y + 1 - 2 &= 9 + 1 \\ (x-3)^2 + (y+1)^2 &= 12 \end{aligned}$$

Thus, the circle is located at $\boxed{\text{center } (3, -1)}$ with $\boxed{\text{radius } 2\sqrt{3} \text{ units}}$.

3.2b Problem: $x^2 + 4x + y^2 + 6y + 9 = 0$.

3.2b Solution: Completing the square for the x and y component, we can form the equation for a circle:

$$\begin{aligned} x^2 + 4x + y^2 + 6y + 9 &= 0 \\ x^2 + 4x + 4 + (y+3)^2 &= 4 \\ (x+2)^2 + (y+3)^2 &= 4 \end{aligned}$$

Thus, the circle is located at $\boxed{\text{center } (-2, -3)}$ with $\boxed{\text{radius 2 units}}$.

3.2c Problem: $x^2 + \frac{1}{3}x + y^2 - \frac{10}{3}y = \frac{127}{9}$.

3.2c Solution: Completing the square for the x and y component, we can form the equation for a circle:

$$\begin{aligned} x^2 + \tfrac{1}{3}x + y^2 - \tfrac{10}{3}y &= \tfrac{127}{9} \\ x^2 + \tfrac{1}{3}x + \tfrac{1}{36} + y^2 - \tfrac{10}{3}y + \tfrac{25}{9} &= \tfrac{127}{9} + \tfrac{1}{36} + \tfrac{25}{9} \\ \left(x + \tfrac{1}{6}\right)^2 + \left(y - \tfrac{5}{3}\right)^2 &= \tfrac{203}{12} \end{aligned}$$

Thus, the circle is located at $\boxed{\text{center } (-\frac{1}{6}, \frac{5}{3})}$ with $\boxed{\text{radius } \sqrt{\frac{203}{12}} \text{ units}}$.

3.2d Problem: $x^2 + y^2 = \frac{3}{2}x - y + \frac{35}{16}$.

3.2d Solution: Completing the square for the x and y component, we can form the equation for a circle:

$$x^2 + y^2 = \tfrac{3}{2}x - y + \tfrac{35}{16}$$
$$x^2 - \tfrac{3}{2}x + y^2 + y = \tfrac{35}{16}$$
$$x^2 - \tfrac{3}{2}x + \tfrac{9}{16} + y^2 + y + \tfrac{1}{4} = \tfrac{35}{16} + \tfrac{9}{16} + \tfrac{1}{4}$$
$$(x - \tfrac{3}{4})^2 + (y + \tfrac{1}{2})^2 = 3$$

Thus, the circle is located at $\boxed{\text{center } (-\tfrac{3}{4}, -\tfrac{1}{2})}$ with $\boxed{\text{radius } \sqrt{3} \text{ units}}$.

3.3 Context: Water is flowing from a major broken water main at the intersection of two streets. The resulting puddle of water is circular and the radius r of the puddle is given by the equation $r = 5t$ feet, where t represents time in seconds elapsed since the the main broke.

3.3a Problem: When the main broke, a runner was located 6 miles from the intersection. The runner continues toward the intersection at the constant speed of 17 feet per second. When will the runner's feet get wet?

3.3a Solution: The runner was initially 6 miles, or $6 \cdot 5280$ feet $= 31,680$ feet from the intersection at 17 feet per second. The puddle approaches the runner at a rate of $5t$. If we are to fix the origin at the location of the puddle, the runner approaches the puddle at a rate of $5 + 17 = 22$ feet per second, or the sum of their two speeds. Since the two are 6 miles apart, the solution t is the amount of time it takes the runner, moving at 22 ft/sec, to cover that distance. Thus, the following equation can be constructed:

$$31,680 = (5 + 17)t$$
$$31,680 = 22t$$
$$t = 1440 \text{ seconds} \rightarrow \tfrac{1440}{60} = 24 \text{ minutes}$$

Therefore, the runner gets his feet wet at $\boxed{\text{24 minutes.}}$

3.3b Problem: Suppose, instead, that when the main broke, the runner was 6 miles east, and 5000 feet north of the intersection. The runner runs due west at 17 feet per second. When will the runner's feet get wet?

3.3b Solution: The equation of the circumference of the puddle can be represented as $x^2 + y^2 = (5t)^2 = 25t^2$. The location of the runner is $(6 \text{ miles} - 17t \text{ feet})^2 + (5,000 \text{ feet})^2 = 25t^2$. Solving for t for an appropriate time in which the position of the runner falls on the circumference of the puddle:

$$289t^2 - 1,077,120t + 1,028,622,400 = 25t^2$$
$$264t^2 - 1,077,120t + 1,028,644,400 = 0$$
$$t = \frac{1,077,120t \pm \sqrt{(1,077,120)^2 - 4(264)(1,028,644,400)}}{2(264)}$$
$$t = \frac{1,077,120t \pm \sqrt{73962240000}}{528}$$
$$t = 2040 \pm \frac{50}{33}\sqrt{115566}$$
$$t \approx 1524.9242 \text{ or } 2555.0758 \text{ seconds}$$
$$t \approx 25.4154 \text{ or } 42.5846 \text{ minutes}$$

When the latter solution is plugged back into the equation, the result is incorrect:

$$264(2,555.0758)^2 - 1,077,120(2,555.0758) + 1,028,644,400 \approx 22,013.05 \neq 0$$

Therefore, the former is the correct solution. The runner's feet gets wet after $\boxed{\approx 25.4154 \text{ minutes.}}$