Collingwood 51

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14.2 still needs to be completed.

Problem 14.1

Context: Give the domain of each of the following functions. Find the $x-$ and $y-$ intercepts of each function. Sketch a graph and indicate any vertical or horizontal asymptotes. Give equations for the asymptotes.

Part A Problem: $f(x) = \frac{2x}{x-1}$

Part A Solution: Let us begin by finding the asymptotes. $f(x)$ is already in the desired form, so there is no need to simplify. It follows that the equations of the asymptotes are $y = 2$ and $x = 1$. Thus, the domain is $(\infty, 1) \cup (1, \infty)$.

The y-intercept occurs when $x = 0$, yielding $\frac{2(0)}{\text{arbitrary value}} = 0$; thus, the y-intercept occurs at $y = 0$.

The x-intercept can be solved for as follows:

$$\frac{2x}{x-1} = 0$$
$$2x = 0$$
$$x = 0$$

Thus, the x-intercept occurs at $x = 0$.

Part B Problem: $g(x) = \frac{3x+2}{2x-5}$

Part B Solution: Let us begin by finding the asymptotes. $g(x)$ can be simplified as such:

$$\frac{3x+2}{2x-5} = \frac{3x+2}{2\left(x-\frac{5}{2}\right)} = \frac{\frac{3}{2}x+1}{x-\frac{5}{2}}$$

It follows that the equations of the asymptotes are $y = \frac{3}{2}$ and $x = \frac{5}{2}$. Thus, the domain is $\left(\infty, \frac{5}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$.

The y-intercept occurs when $x = 0$, yielding $\frac{3(0)+2}{2(0)-5} = -\frac{2}{5}$; thus, the y-intercept occurs at $y = -\frac{2}{5}$.

The x-intercept can be solved for as follows:

$$\frac{3x+2}{2x-5} = 0$$
$$3x + 2 = 0$$
$$x = -\frac{2}{3}$$

Thus, the x-intercept occurs at $x = -\frac{2}{3}$.

Part C Problem: $h(x) = \frac{x+1}{x-2}$

Part C Solution: Let us begin by finding the asymptotes. $h(x)$ is already in the desired form, so there is no need to simplify. It follows that the equations of the asymptotes are $y = 1$ and $x = 2$. Thus, the domain is $(\infty, 2) \cup (2, \infty)$.

The y-intercept occurs when $x = 0$, yielding $\frac{0+1}{0-2} = -\frac{1}{2}$; thus, the y-intercept occurs at $y = -\frac{1}{2}$.

The x-intercept can be solved for as follows:

$$\frac{x+1}{x-2} = 0 = 0$$
$$x = -1$$

Thus, the x-intercept occurs at $x = -1$.

Part D Problem: $j(x) = \frac{4x-12}{x+8}$

Part D Solution: Let us begin by finding the asymptotes. $j(x)$ is already in the desired form, so there is no need to simplify. It follows that the equations of the asymptotes are $y = 4$ and $x = -8$. Thus, the domain is $(\infty, -8) \cup (-8, \infty)$.

The y-intercept occurs when $x = 0$, yielding $\frac{4(0)-12}{0+8} = -\frac{3}{2}$; thus, the y-intercept occurs at $y = -\frac{3}{2}$.

The x-intercept can be solved for as follows:

$$\frac{4x-12}{x+8} = 0$$
$$4x = 12$$
$$x = 3$$

Thus, the x-intercept occurs at $x = 3$.

Part E Problem: $k(x) = \frac{8x+16}{5x-\frac{1}{2}}$

Part E Solution: Let us begin by finding the asymptotes. $k(x)$ can be simplified as such:

$$\frac{8x+16}{5x-\frac{1}{2}} = \frac{8x+16}{5\left(x-\frac{1}{10}\right)} = \frac{\frac{8}{5}x+\frac{16}{5}}{x-\frac{1}{10}}$$

It follows that the equations of the asymptotes are $y = \frac{8}{5}$ and $x = \frac{1}{10}$. Thus, the domain is $\left(\infty, \frac{1}{10}\right) \cup \left(\frac{1}{10}, \infty\right)$.

The y-intercept occurs when $x = 0$, yielding $\frac{8(0)+16}{5(0)-\frac{1}{2}} = -32$; thus, the y-intercept occurs at $y = -32$.

The x-intercept can be solved for as follows:

$$\frac{8x+16}{5x-\frac{1}{2}} = 0$$
$$8x + 16 = 0$$
$$x = -2$$

Thus, the x-intercept occurs at $x = -2$.

Part F Problem: $m(x) = \frac{9x+24}{35x-100}$

Part F Solution: Let us begin by finding the asymptotes. $k(x)$ can be simplified as such:

$$\frac{9x+24}{35x-100} = \frac{9x+24}{35\left(x-\frac{100}{35}\right)} = \frac{\frac{9}{35}x+\frac{24}{35}}{x-\frac{100}{35}}$$

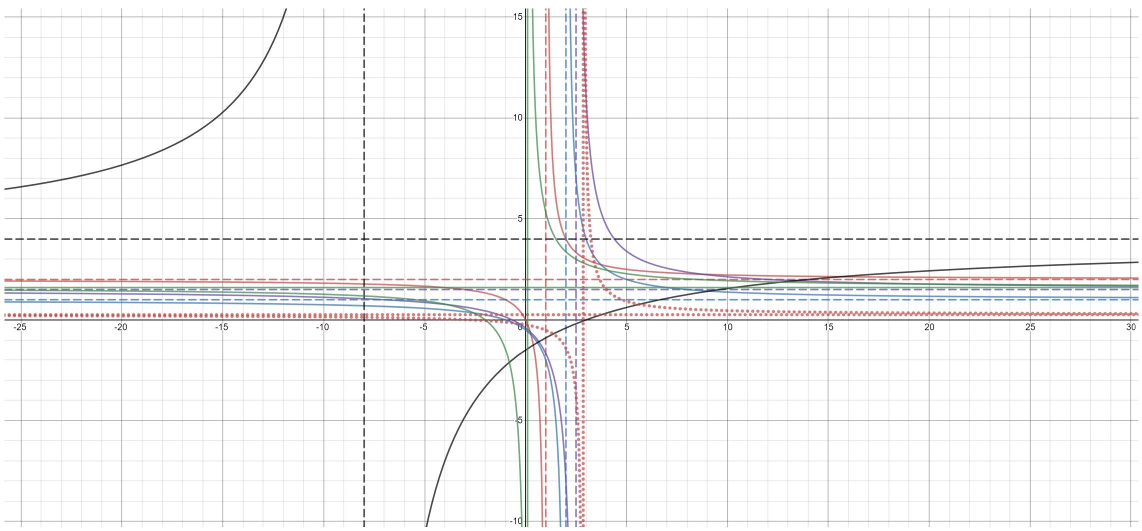
It follows that the equations of the asymptotes are $y = \frac{9}{35}$ and $x = \frac{100}{35}$. Thus, the domain is $\left(\infty, \frac{100}{35}\right) \cup \left(\frac{100}{35}, \infty\right)$.

The y-intercept occurs when $x = 0$, yielding $\frac{9(0)+24}{35(0)-100} = -\frac{6}{25}$; thus, the y-intercept occurs at $y = -\frac{6}{25}$. The x-intercept can be solved for as follows:

$$\frac{9x+24}{35x-100} = 0$$
$$9x = -24$$
$$x = -\frac{24}{9}$$

Thus, the x-intercept occurs at $x = -\frac{24}{9}$.

Graphs for Parts A-F: $f(x)$ in red, $g(x)$ in purple, $h(x)$ in blue, $j(x)$ in black, $k(x)$ in green, $m(x)$ in dotted-red.



Problem 14.2

Context: Oscar is hunting magnetic fields with his Gauss meter, a device for measuring the strength and polarity of magnetic fields. The reading on the meter will increase as Oscar gets closer to a magnet. Oscar is in a long hallway at the end of which is a room containing an extremely strong magnet. When he is far down the hallway from the room, the meter reads a level of 0.2. He then walks down the hallway and enters the room. When he has gone 6 feet into the room, the meter reads 2.3. Eight feet into the room, the meter reads 4.4.

Part A Problem: Give a linear-to-linear rational model relating the meter reading y to how many feet x Oscar has gone into the room.

Part A Solution: The linear-to-linear rational model is of form $y = \frac{Ax-B}{x+C}$. Because we know that the horizontal asymptote is $y = 0.2$, we can then solve for B and C in $y = \frac{0.2x-B}{x+C}$ given the points $(6, 2.3)$ and $(8, 4.4)$.

$$2.3 = \frac{0.2(6)-B}{6+C} \rightarrow B = -2.3C - 12.6$$

$$4.4 = \frac{0.2(8)-B}{8+C} \rightarrow B = -4.4C - 33.6$$

Thus, setting the two values of B equal to each other, we have:

$$-2.3C - 12.6 = -4.4C - 33.6 \rightarrow -10$$

Thus, $B = -2.3(-10) - 12.6 = 10.4$. Thus, the model is $y = \frac{0.2x-10.4}{x-10}$.

Part B Problem: How far must he go for the meter to reach 10? 100?

Part B Solution: Solving for when the meter reaches 10:

$$\frac{0.2x - 10.4}{x - 10} = 10 \rightarrow 2x - 104 = 100x - 1000 \rightarrow x = \frac{64}{7}$$

Thus, Oscar must walk $\frac{64}{7}$ feet into the room for the meter to read 10.

Solving for when the meter reaches 100:

$$\frac{0.2x - 10.4}{x - 10} = 100 \rightarrow 2x - 104 = 1000x - 1000 \rightarrow x = \frac{4948}{499}$$

Thus, Oscar must walk $\frac{4948}{499}$ feet into the room for the meter to read 100.

Part C Problem: Considering your function from part (a) and the results of part (b), how far into the room do you think the magnet is?

Part C Solution: Somewhere very close to 10 feet, since this should be when the Gauss meter yields the highest value, indicating it is closest. This is the vertical asymptote of the function.

Problem 14.3

Context: In 1975 I bought an old Martin ukulele for 200 dollars. In 1995 a similar uke was selling for 900 dollars. In 1980 I bought a new Kamaka uke for 100 dollars. In 1990 I sold it for 400 dollars.

Part A Problem: Give a linear model relating the price p of the Martin uke to the year t. Take $t = 0$ in 1975.

Part A Solution: We have two points: $(0, 200)$ and $(20, 900$; the slope is $\frac{700}{20} = 35$. The model is thus $p = 35t + 200$.

Part B Problem: Give a linear model relating the price q of the Kamaka uke to the year t. Again take $t = 0$ in 1975.

Part B Solution: We have two points: $(5, 100)$ and $(15, 400)$; the slope is $\frac{300}{10} = 30$. The model is thus $q = 30(t - 5) + 100$.

Part C Problem: When is the value of the Martin twice the value of the Kamaka?

Part C Solution: We have $p = 2q$; substituting p and q with functions of t, then solving for t yields:

$$\begin{aligned}
35t + 200 &= 2(30(t-5) + 100) \\
35t + 200 &= 60(t-5) + 200 \\
35t &= 60(t-5) \\
\frac{7}{12}t &= t - 5 \\
\frac{5}{12}t &= 5 \\
t &= 12
\end{aligned}$$

12 years past 1975 is 1987; thus, the value of the Martin is twice that of the Kamaka in 1987.

Part D Problem: Give a function $f(t)$ which gives the ratio of the price of the Martin to the price of the Kamaka.

Part D Solution: This is simply $\frac{p}{q}$; substituting yields $f(t) = \frac{35t+200}{30(t-5)+100}$

Part E Problem: In the long run, what will be the ratio of the prices of the ukuleles?

Part E Solution: To find the horizontal asymptote, first we must simplify $f(t)$ as follows:

$$\frac{35t+200}{30(t-5)+100} = \frac{35t+200}{30t-50} = \frac{35t+200}{30\left(t-\frac{5}{3}\right)} = \frac{\frac{7}{6}t+\frac{20}{3}}{t-\frac{5}{3}}$$

It follows that the horizontal asymptote is given by $y = \frac{7}{6}$. Thus, "in the long run", the ratio of the prices of ukuleles will be $7:6$.

Problem 14.4

Context: Isobel is producing and selling cassette tapes of her rock band. When she had sold 10 tapes, her net profit was 6 dollars. When she had sold 20 tapes, however, her net profit had shrunk to 4 dollars due to increased production expenses. But when she sold 30 tapes, her net profit had rebounded to 8 dollars.

Part A Problem: Give a quadratic model relating Isobel's net profit y to the number of tapes sold x.

Part A Solution: We have three points: $(10,6)$, $(20,4)$, and $(30,8)$. We can thus write three equations:

$$6 = 100a + 10b + c$$

$$4 = 400a + 20b + c$$

$$8 = 900a + 30b + c$$

Let us write this as a matrix:

$$\begin{pmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 900 & 30 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix}$$

Solving (with a matrix calculator):

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 900 & 30 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{3}{100} \\ -\frac{11}{10} \\ 14 \end{pmatrix}$$

Thus, our quadratic model is $y = \frac{3}{100}x^2 - \frac{11}{10}x + 14$.

Part B Problem: Divide the profit function in part a by the number of tapes sold x to get a model relating average profit w per tape to the number of tapes sold.

Part B Solution: This yields $y = \frac{\frac{3}{100}x^2 - \frac{11}{10}x + 14}{x}$

Part C Problem: How many tapes must she sell in order to make 1.2 per tape in net profit?

Part C Solution: We have that

$$\frac{\frac{3}{100}x^2 - \frac{11}{10}x + 14}{x} = 1.2$$

Solving yields:

$$\begin{aligned}\frac{\frac{3}{100}x^2-\frac{11}{10}x+14}{x}&=1.2\\ \frac{3x^2-110x+1400}{100x}&=1.2\\ 3x^2-110x+1400&=120x\end{aligned}$$

$$\begin{aligned}x&=\frac{-(-230)\pm\sqrt{(-230)^2-4\cdot 3\cdot 1400}}{2\cdot 3}\\ &=70,\frac{20}{3}\end{aligned}$$

Hence, she can sell 70 or $\frac{20}{3}$ tapes to make 1.20 dollars in net profit for each, although selling $\frac{20}{3}$ tapes may not be practical, in which case only 70 tapes may make more sense.

Problem 14.5

Problem: Find the linear-to-linear function whose graph passes through the points (1,1), (5,2) and (20,3). What is its horizontal asymptote?

Solution: We have that $f(x)=\frac{Ax+B}{x+C}$. Then, we have three equations:

$$\begin{aligned}\frac{A+B}{1+C}&=1\to A+B=1+C\to A+B-C=1\\ \frac{5A+B}{5+C}&=2\to 5A+B=10+2C\to 5A+B-2C=10\\ \frac{20A+B}{20+C}&=3\to 20A+B=60+3C\to 20A+B-3C=60\end{aligned}$$

Writing this in matrix form:

$$\begin{pmatrix}1&1&-1\\5&1&-2\\20&1&-3\end{pmatrix}\cdot\begin{pmatrix}A\\B\\C\end{pmatrix}=\begin{pmatrix}1\\10\\60\end{pmatrix}$$

$$\begin{pmatrix}A\\B\\C\end{pmatrix}=\begin{pmatrix}1&1&-1\\5&1&-2\\20&1&-3\end{pmatrix}^{-1}\cdot\begin{pmatrix}1\\10\\60\end{pmatrix}=\begin{pmatrix}\frac{41}{11}\\\frac{35}{11}\\\frac{65}{11}\end{pmatrix}$$

The model is thus $f(x)=\frac{\frac{41}{11}x+\frac{35}{11}}{x+\frac{65}{11}}$. From this equation, we can derive that the horizontal asymptote is $x=\frac{41}{11}$.

Problem 14.6

Problem: Find the linear-to-linear function whose graph has $y=6$ as a horizontal asymptote and passes through $(0,10)$ and $(3,7)$.

Solution: We know that the graph has form $y=\frac{6x+B}{x-C}$, and that it passes through $(0,10)$ and $(3,7)$. We have, then, two equations:

$$10=-\frac{B}{C}\to B=-10C$$

$$7=\frac{18+B}{3-C}$$

Plugging the first equation into the second yields

$$7=\frac{18-10C}{3-C}$$

Solving:

$$\begin{aligned}18 - 10C &= 21 - 7C \\ -3C &= 3 \\ C &= -1\end{aligned}$$

It follows that $B = 10$. The linear-to-linear function is thus $y = \frac{6x+10}{x+1}$.

Problem 14.7

Problem: The more you study for a certain exam, the better your performance on it. If you study for 10 hours, your score will be 65 percent. If you study for 20 hours, your score will be 95 percent. You can get as close as you want to a perfect score just by studying long enough. Assume your percentage score is a linear-to linear function of the number of hours that you study. If you want a score of 80 percent, how long do you need to study?

Solution: We have two points - $(10, 65)$ and $(20, 95)$, with a horizontal asymptote of $y = 100$ – a perfect score. Using this information, we can construct the model with form $y = \frac{100x+B}{x+C}$. We thus have that:

$$65 = \frac{100(10) + B}{10 + C} \rightarrow 650 + 65C = 1000 + B \rightarrow B = 65C - 350$$

$$95 = \frac{100(20) + B}{20 + C} \rightarrow 1900 + 95C = 2000 + B \rightarrow B = 95C - 100$$

Thus,

$$65C - 350 = 95C - 100 \rightarrow -30C = 250 \rightarrow C = -\frac{25}{3}$$

Then,

$$B = 65\left(-\frac{25}{3}\right) - 350 = -\frac{2675}{3}$$

Thus, the model is $y = \frac{100x - \frac{2675}{3}}{x - \frac{25}{3}}$. We can then solve for x when the model equals 80.

$$\begin{aligned}80 &= \frac{100x - \frac{2675}{3}}{x - \frac{25}{3}} \\ 80x - \frac{2000}{3} &= 100x - \frac{2675}{3} \\ 20x &= \frac{675}{3} \\ x &= \frac{675}{60}\end{aligned}$$

Thus, if we want a score of 80 percent, you should probably study for $\frac{675}{60}$ hours.

Problem 14.8

Context: A street light is 10 feet above a straight bike path. Olav is bicycling down the path at a rate of 15 MPH. At midnight, Olav is 33 feet from the point on the bike path directly below the street light. (See the picture). The relationship between the intensity C of light (in candlepower) and the distance d (in feet) from the light source is given by $C = \frac{k}{d^2}$, where k is a constant depending on the light source.

Part A Problem: From 20 feet away, the street light has an intensity of 1 candle. What is k?

Part A Solution: We have that $1 = \frac{k}{400} \rightarrow k = 400$.

Part B Problem: Find a function which gives the intensity of the light shining on Olav as a function of time, in seconds.

Part B Solution: Olav moves at 15 MPH, which is equivalent to
15 MPH is the same as $15 \cdot 60^2$ MPS, which is equivalent to $\frac{15 \cdot 5280}{60^2} = 22$ FPS. The distance of Olav from the street light in terms of t is $d(t) = \sqrt{(33-22t)^2 + 10^2}$. Thus, the function that gives the intensity of the light shining by plugging $d(t)$ in for d yields

$$C(t) = \frac{400}{(33-22t)^2 + 10^2}$$

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Part C Problem: When will the light on Olav have maximum intensity?

Part C Solution: As the distance between Olav and the lamp decreases, the brightness increases monotonically. Thus, the smallest distance yields the most intense light. The smallest distance between Olav and the lamp is 10 feet. We can solve for the time t at which the distance is 10 feet.

$$\begin{aligned}\sqrt{(33-22x)^2 + 10^2} &= 10 \\ (33-22x)^2 + 10^2 &= 100 \\ 33 - 22x &= 0 \\ x &= \frac{3}{2}\end{aligned}$$

Thus, the most intense brightness occurs at $\frac{3}{2}$ seconds. This is wrong and needs to be fixed.

Part D Problem: When will the intensity of the light be 2 candles?

Part D Solution: We can solve for when $C(t) = 2$:

$$\begin{aligned}\frac{400}{(33-22x)^2 + 10^2} &= 2 \\ (33-22x)^2 + 10^2 &= 200 \\ 33 - 22x &= \pm 10 \\ x &= \frac{\pm 10 - 33}{-22} = \frac{23}{22} \text{ or } \frac{43}{22} \text{ seconds}\end{aligned}$$

Problem 14.9

Context: For each of the following find the linear to linear function $f(x)$ satisfying the given requirements:

Part A Problem: $f(0) = 0, f(10) = 10, f(20) = 15$.

Part A Solution: We have the points $(0,0)$, $(10,10)$, and $(20,15)$. Given a model in the form $y = \frac{Ax+B}{x+C}$, three equations can be written:

$$0 = \frac{B}{C} \to B = 0$$

$$10 = \frac{10A+B}{10+C} \to 10 = \frac{10A}{10+C} \to 100 + 10C = 10A \to 10A - 10C = 100 \to 20A - 20C = 200$$

$$15 = \frac{20A+B}{20+C} \to 300 + 15C = 20A \to 20A - 15C = 300$$

Subtracting the second from the third yields $5C = 100 \to C = 20$. Thus, the $f(x) = \frac{30x}{x+20}$.

Part B Problem: $f(0) = 10, f(5) = 4, f(20) = 3$

Part B Solution: We have the three points $(0, 10)$, $(5, 4)$, and $(20, 3)$. Given a model in the form $y = \frac{Ax+B}{x+C}$, three equations can be written:

$$10 = \frac{B}{C} \rightarrow B = 10C$$

$$4 = \frac{5A + 10C}{5 + C} \rightarrow 20 + 4C = 5A + 10C \rightarrow 6C + 5A = 20 \rightarrow 24C + 20A + 80$$

$$3 = \frac{20A + B}{20 + C} \rightarrow 20A + 10C = 60 + 3C \rightarrow 20A + 7C = 60$$

Solving by subtracting the third equation from the second yields $17C = 20 \rightarrow C = \frac{20}{17}$; thus $B = \frac{200}{17}$. Furthermore, we have that $20A + 7\left(\frac{20}{17}\right) = 60 \rightarrow A = \frac{44}{17}$. Thus, the model is $\frac{\frac{44}{17}x + \frac{200}{17}}{x + \frac{20}{17}}$.

Part C Problem: $f(10) = 20, f(30) = 25$, and the graph of $f(x)$ has $y = 30$ as its horizontal asymptote.

Part C Solution: We have the points $(10, 20)$ and $(30, 25)$, with the model $\frac{30x+B}{x+C}$. Thus, there are two linear equations:

$$20 = \frac{300 + B}{10 + C} \rightarrow 200 + 20C = 300 + B \rightarrow 20C - B = 100$$

$$25 = \frac{900 + B}{30 + C} \rightarrow 750 + 25C = 900 + B \rightarrow 25C - B = 150$$

Subtracting the second from the third yields $5C = 50 \rightarrow C = 10$, and thus $200 - B = 100 \rightarrow B = 100$. The function is thus $f(x) = \frac{30x+100}{x+10}$.

Problem 14.10

Context: The number of customers in a local dive shop depends on the amount of money spent on advertising. If the shop spends nothing on advertising, there will be 100 customers/day. If the shop spends 100 dollars, there will be 200 customers/day. As the amount spent on advertising increases, the number of customers/day increases and approaches (but never exceeds) 400 customers/day.

Part A Problem: Find a linear to linear rational function $y = f(x)$ that calculates the number y of customers/day if x dollars is spent on advertising.

Part A Solution: The asymptote is $y = 400$. The two data points given are $(0, 100)$ and $(100, 200)$. Thus, we attempt to find the parameters A and B in the model $y = \frac{400x+B}{x+C}$. We have that $\frac{B}{C} = 100 \rightarrow B = 100C$ and that $200 = \frac{40000+B}{100+C} \rightarrow 20000 + 200C = 40000 + B \rightarrow 200C - B = 20000$. Substituting $100C$ for B yields

$$100C = 20000 \rightarrow C = 200 \rightarrow C = 20000$$

The model is thus $f(x) = \frac{400x+20000}{x+200}$.

Part B Problem: How much must the shop spend on advertising to have 300 customers/day?

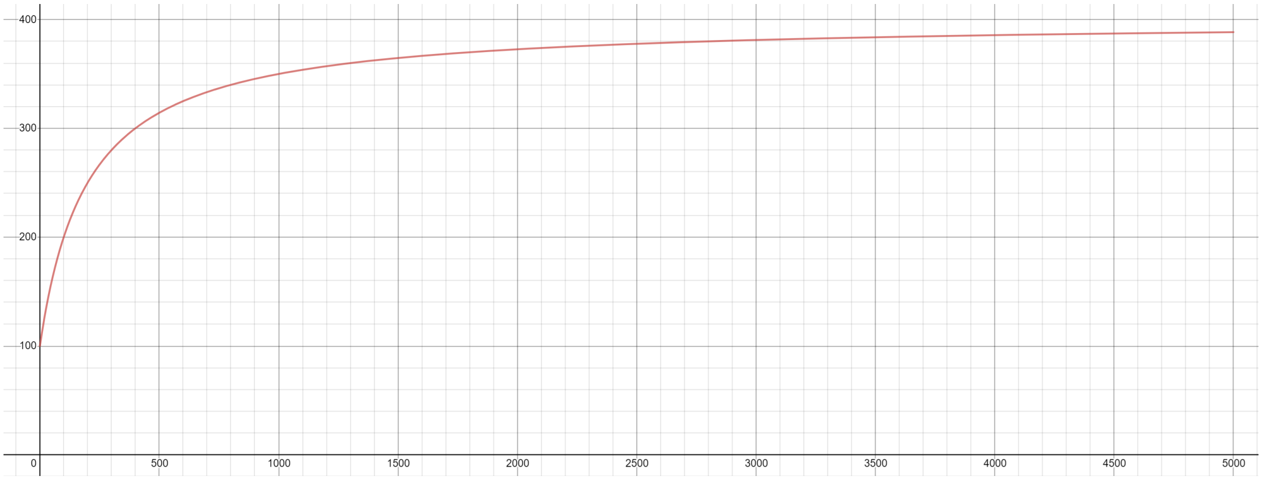
Part B Solution:

$$\frac{400X + 20000}{X + 200} = 300$$
$$400x + 20000 = 60000 + 300X$$
$$100X = 40000$$
$$X = 400$$

The shop must spend 400 dollars on advertising to have 300 customers/day.

Part C Problem: Sketch the graph of the function $y = f(x)$ on the domain $0 \leq x \leq 5000$.

Part C Solution:



Part D Problem: Find the rule, domain and range for the inverse function from part (c). Explain in words what the inverse function calculates.

Part D Solution: $f(x)$ calculates the customers per day given the amount of money spent on advertising. Thus, $f^{-1}(x)$ calculates the amount of money spent on advertising given the customers per day. Solving for the inverse:

$$x = \frac{400y + 20000}{y + 200}$$
$$yx + 200x = 400y + 20000$$
$$yx - 400y = -200x + 20000$$
$$y(-(400 - x)) = -(200x - 20000)$$
$$y = \frac{200x - 20000}{400 - x}$$

The inverse is this $f^{-1}(x) = \frac{200x-20000}{400-x}$. The range is the domain of $f(x)$, which is $[0, 5000]$. The domain is the range of $f(x)$, which is $\left[100, f^{-1}(5000)\right)$.