Collingwood 50

Andre Ye

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Problem 13.6

Context: An *isosceles* triangle has sides of length x, x, and y. In addition, assume the triangle has perimeter 12.

Part A Problem: Find the rule for a function that computes the area of the triangle as a function of x. Describe the largest possible domain of this function.

Part A Solution: If we envision a right triangle with base of length $\frac{y}{2}$ and hypotenuse of length x, we have the height to be $\sqrt{x^2 - \frac{y^2}{4}}$. Given x, we know that $y = 12 - 2x$ The area of two of these right triangles, which together make the isosceles triangle, make area $a(x) = \frac{(12-2x)}{2} \cdot \sqrt{x^2 - \frac{(12-2x)^2}{4}}$.

We can break $a(x)$ into two components. The first, $\frac{(12-2x)}{2}$, suggests that x cannot be larger than 6, or else the component will be negative, causing the entire output of $a(x)$ to be negative as the square root expression cannot yield a negative number. The second, $\sqrt{x^2 - \frac{(12-2x)^2}{4}}$, requires a more involved process. For the expression under the square root to be valid, it must be true that $x^2 - \frac{(12-2x)^2}{4} \geq 0$. Simplifying yields $12\,(x-3) \geq 0 \to x \geq 3$.

Combining these two derived conditions, we have the domain of $a(x)$ to be $[3, 6]$.

Part B Problem: Assume that the maximum value of the function $a(x)$ occurs when $x = 4$. Find the maximum value of $z = a(x)$ and $z = 2x(3x+3)+1$.

Part B Solution: In the first case, the maximum value of $a(x)$ is at $x = 4$, as given to us by the problem; plugging this into $a(x)$ yields a maximum value of $4\sqrt{3}$. In the second case, let us rewrite $2a(3x+3)+1$ as $2a(3(x+1))+1$; therefore, we have that the following operations were done:

1. Horizontal shrink by a factor of 3.
2. Horizontal shift by 1 unit to the left.
3. Vertical stretch by a factor of 2.
4. Vertical shift by 1 unit upwards.

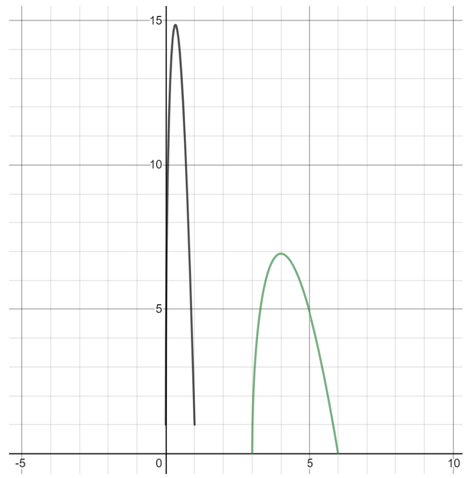
Applying this to the maximum area of the triangle (as this is the output of $a(x)$ – part of the y-axis, to use terminology loosely - only vertical transformations apply) yields $2(4\sqrt{3})+1 \to 16\sqrt{3}+1$. The maximum value of $2a(3x+3)+1$ is hence $8\sqrt{3}+1$.

Part C Problem: The graph of $z = a(x)$ from part (a) is shown below. Sketch the graph and find the rule for the function $2a(3x+3)+1$; make sure to specify the domain and range of this new function.

Part C Solution: The transformations (rules) for this function were listed out earlier. Hence, we have that

- The domain of $a(x)$ is $[3, 6]$ (as found in part a). Applying horizontal transformations yields a domain of $[0, 1]$.
- The range of $a(x)$ is $[0, 4\sqrt{3}]$. Applying vertical transformations yields a range of $[1, 8\sqrt{3} + 1]$.

The graph is



Problem 13.7

Context: Describe how each graph differs from that of $y = x^2$.

Parts A-E Problems and Answers: Answer is provided in the form of transformations, in the order given. That is, the answer provides how to arrive at the equation given by transforming $y = x^2$ in the order of the transformations given. 'V.' refers to vertical, 'H.' refers to horizontal.

Part	Problem	Answer
a	$y = 2x^2$	V. stretch by factor of 2.
b	$y = x^2 - 5$	V. shift down 5 units.
c	$y = (x - 4)^2$	H. shift right 4 units.
d	$y = (3x - 12)^2$	H. shift right 12 units, H. compression by factor of 3.
e	$y = 2(3x - 12)^2 - 5$	H. shift right 12 units, H. compression by factor of 3, V. stretch by factor of 2, V. shift down 5 units.

Problem 13.8

Context: In each case, start with the function $y = |x|$ and perform the operations described to the graph, *in the order specified*. Write out the resulting rule for the function and sketch the final graph you gain.

Part A Problem:

1. Horizontally compress by a factor of 2,
2. Horizontally shift to the left by 2,
3. Vertically stretch by a factor of 7,
4. Vertically shift up 2 units.

Part A Solution: Applying these transformations one by one yields the following process:

$$|x| \to |2x| \to |2(x+2)| \to 7|2(x+2)| \to 7|2(x+2)|+2$$

Part B Problem:

1. Horizontally stretch by a factor of 2,
2. Horizontally shift to the left by 2,
3. Vertically compress by a factor of 7,
4. Vertically shift down 2 units.

Part B Solution: Applying these transformations one by one yields the following process:

$$|x| \to \left|\frac{1}{2}x\right| \to \left|\frac{1}{2}(x+2)\right| \to \frac{1}{7}\left|\frac{1}{2}(x+2)\right| \to \frac{1}{7}\left|\frac{1}{2}(x+2)\right|-2$$

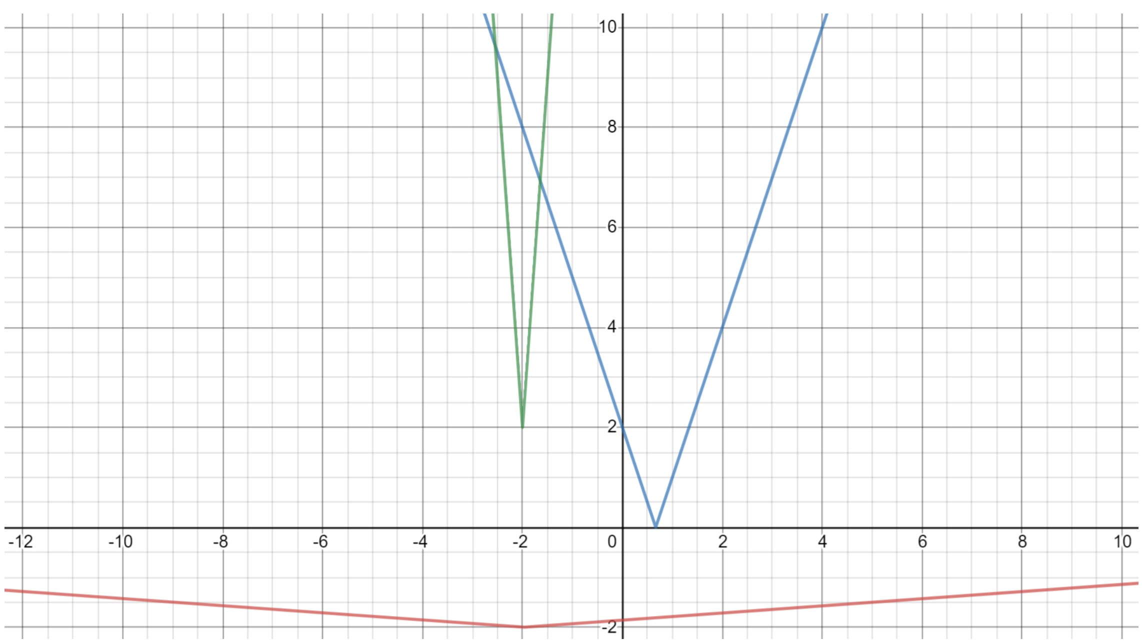
Part C Problem:

1. Horizontally shift to the right 2 units,
2. Horizontally compress by a factor of 3.

Part C Solution: Applying these transformations one by one yields the following process:

$$|x| \to |x-2| \to |3x-2|$$

Graphs for Parts A-C Solutions: Red: $\frac{1}{7}\left|\frac{1}{2}(x+2)\right|-2$. Blue: $|3x-2|$. Green: $7|2(x+2)|+2$.



Problem 13.9

Part A

Part A Context: Begin with the function $y = f(x) = 2^x$.

A1 Problem: Rewrite each of the following functions in standard exponential form.

A1 Solution:

Problem	Work + Answer
$f(2x)$	$2^{2x} \to 4^x$
$f(x-1)$	$2^{x-1} \to \frac{1}{2} \cdot 2^x$
$f(2x-1)$	$2^{2x-1} \to \frac{1}{2} \cdot 4^x$
$f(2(x-1))$	$2^{2(x-1)} \to 4^{(x-1)} \to \frac{1}{4} \cdot 4^x$
$3f(x)$	$3 \cdot 2^x$
$3f(2(x-1))$	$3 \cdot 2 \cdot 2^x \to \frac{1}{4} \cdot 3 \cdot 4^x \to \frac{3}{4} \cdot 4^x$

Part A2 Problem: Is the function $3f(2(x-1)) + 1$ a function of exponential type?

Part A2 Solution: We find this to be equal to $\frac{3}{4} \cdot 4^x \to \frac{3}{4} \cdot 4^x + 1$; as this cannot be written in the exponential form ab^x – as the $+1$ addition cannot be incorporated into either the a or the b parameters – and hence is not a function of exponential type.

Part A3 Problem: Sketch out graphs in the same coordinate system and explain which graphical operations have been carried out.

Part A3 Solution: Explanations are given as follows:
From $f(x)$ to $f(x)$ –

1. No transformations are required.

From $f(x)$ to $f(2x)$ –

1. Horizontal compression by a factor of 2.

From $f(x)$ to $f(2(x-1))$ –

1. Horizontal compression by a factor of 2;
2. Horizontal shift right by 1 unit.

From $f(x)$ to $3f(2(x-1))$ –

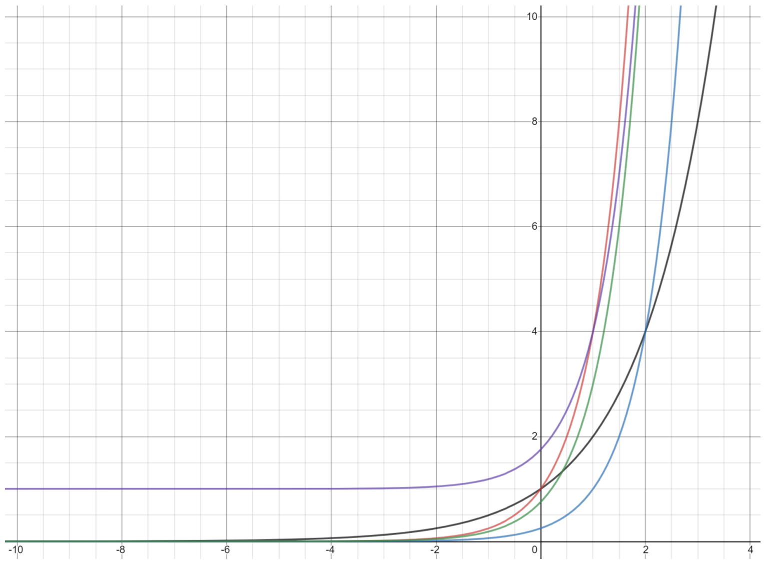
1. Horizontal compression by a factor of 2;
2. Horizontal shift right by 1 unit;
3. Vertical stretch by a factor of 3.

From $f(x)$ to $3f(2(x-1)) + 1$ –

1. Horizontal compression by a factor of 2;
2. Horizontal shift right by 1 unit;
3. Vertical stretch by a factor of 3;

4. Vertical shift up by 1 unit.

Graphs are displayed below. Black: 2^x; red: $f(2x)$; blue: $f(2(x-1))$; green: $3f(2(x-1))$; purple: $3f(2(x-1))+1$



Part B

Part B Problem: In general, explain what happens when you apply the four construction tools to the standard exponential model $y = A_0 b^x$. For which of the four operations is the resulting function still a standard exponential model?

Part B Solution:

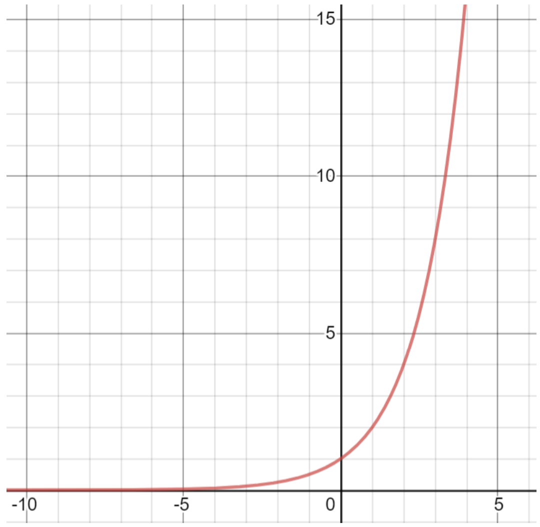
- Horizontal shift by γ units can be expressed as $A_0 b^{x-\gamma}$; this can be simplified as $\frac{A_0}{b^\gamma} \cdot b^x$, which is still in exponential form.
- Horizontal compression/dilation by a factor of γ can be expressed as $A_0 b^{\gamma x}$; this can be simplified as $(A_0 b^\gamma)^x$, which is still in exponential form.
- Vertical shift by γ units can be expressed as $A_0 b^x + \gamma$; this cannot be expressed in exponential form.
- Vertical compression/dilation by a factor of γ can be expressed as $\gamma A_0 b^x$, which is already in exponential form.

Therefore, only vertical shift does not allow for the resulting function to still be a standard exponential model.

Problem 13.10

Problem: Begin with a sketch of the graph of the function $y = 2^x$ on the domain of all real numbers. Describe how to use the "four tools" of Chapter 13 to obtain the graphs of the following functions.

Solution: The sketch of the graph of $y = 2^x$ is as follows:



To obtain -2^x from 2^x: Reflect over the x axis.
To obtain 2^{-x} from 2^x: Reflect over the y axis.
To obtain $3(2^x)$ from 2^x: Expand vertically by a factor of 3.
To obtain $\frac{1}{3}(2^x)$ from 2^x: Shrink vertically by a factor of 3.
To obtain $3 + 2^x$ from 2^x: Shift vertically by 3 units upwards
To obtain $2^x - 2$ from 2^x: Shift vertically by 2 units downwards.
To obtain 2^{x-2} from 2^x: Shift horizontally by 2 units rightwards.
To obtain 2^{x+2} from 2^x: Shift horizontally by 2 units leftwards.
To obtain 2^{3x} from 2^x: Compress horizontally by a factor of 3.
To obtain $2^{\frac{x}{3}}$ from 2^x: Expand horizontally by a factor of 3.

Additional Problem Set Problem 1

Context: Let $f(x)$ be an unknown function, and let $g(x)$ be the function produced by transforming f in the following ways,in order:

1. Translation to the right by 3 units
2. Translation down by 2 units
3. Vertical dilation by a factor of 3
4. Horizontal dilation by a factor of 4
5. Reflection across the y-axis
6. Translation up by 2 units
7. Translation to the left by 3 units
8. Vertical compression by a factor of 3
9. Horizontal compression by a factor of 4
10. Reflection across the y-axis

Part A Problem: Write a formula for g in terms of f.

Part A Solution: We can transform our function according to the following steps as follows:

$$\begin{aligned} f(x) &\implies f(x-3) \\ &\implies f(x-3)-2 \\ &\implies 3(f(x-3)-2) \\ &\implies 3\left(f\left(\frac{x}{4}-3\right)-2\right) \\ &\implies -3\left(f\left(\frac{x}{4}-3\right)-2\right) \\ &\implies -3\left(f\left(\frac{x}{4}-3\right)-2\right)+2 \\ &\implies -3\left(f\left(\frac{x+3}{4}-3\right)-2\right)+2 \\ &\implies \frac{-3\left(f\left(\frac{x+3}{4}-3\right)-2\right)+2}{3} \\ &\implies \frac{-3\left(f\left(\frac{4x+3}{4}-3\right)-2\right)+2}{3} \\ &\implies -\frac{-3\left(f\left(\frac{4x+3}{4}-3\right)-2\right)+2}{3} \end{aligned}$$

Hence, we have that $g(x) = -\frac{-3\left(f\left(\frac{4x+3}{4}-3\right)-2\right)+2}{3}$.

Part B Problem: If you had to achieve the same end result using only one horizontal dilation or compression, one horizontal translation, one vertical dilation or compression, and one vertical translation,in that order, could you? If so, how? If not, why not?

Part B Solution: Let us first simplify $g(x)$. We can simplify as follows:

$$\begin{aligned} -\frac{-3\left(f\left(\frac{4x+3}{4}-3\right)-2\right)+2}{3} &= -\frac{-3\left(f\left(\frac{4x+3-12}{4}\right)-2\right)+2}{3} \\ &= -\frac{-3\left(f\left(\frac{4x-9}{4}\right)-2\right)+2}{3} \\ &= \frac{3f\left(\frac{4x-9}{4}\right)-8}{3} \\ &= f\left(x-\frac{9}{4}\right)-\frac{8}{3} \end{aligned}$$

We can write this as $f\left(x-\frac{9}{4}\right)-\frac{8}{3}$, which means that the steps are

1. Horizontal translation right by $\frac{9}{4}$;
2. Vertical translation down by $\frac{8}{3}$.

Additional Problem Set Problem 2

Problem: Let $f(x)$ be an unknown one-to-one function, and let $g(x) = 3f(2x-4)+1$. Find an expression for $g^{-1}(x)$ in terms of $f^{-1}(x)$. What transformations could be used to reach g^{-1} from f^{-1}?

Solution:

$$
\begin{aligned}
3f(2x-4)+1 &= y \\
3f(2y-4)+1 &= x \\
3f(2y-4) &= x-1 \\
f(2y-4) &= \frac{x-1}{3} \\
2y-4 &= f^{-1}\left(\frac{x-1}{3}\right) \\
2y &= f^{-1}\left(\frac{x-1}{3}\right)+4 \\
y &= \frac{f^{-1}\left(\frac{x-1}{3}\right)+4}{2}
\end{aligned}
$$

The function can thus be expressed as $g^{-1}(x) = \frac{f^{-1}\left(\frac{x-1}{3}\right)+4}{2}$. This can be done in the order of:

1. Horizontal dilation by a factor of 3;
2. Horizontal shift right by 1 unit;
3. Vertical shift up by 4 units;
4. Vertical compression by a factor of 2.