Collingwood 49

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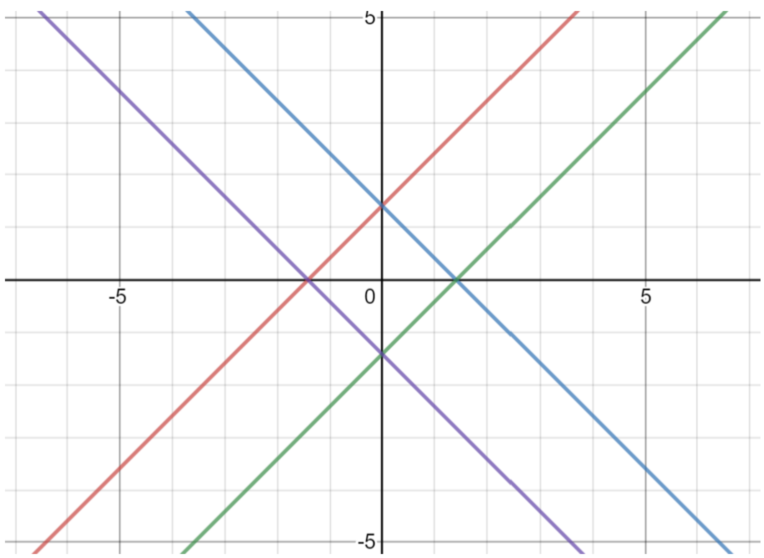
15 February 2021

Problem 13.1

Context: On a single set of axes, sketch a picture of the graphs of the following four equations: $y = -x + \sqrt{2}, y = -x - \sqrt{2}, y = x + \sqrt{2}$, and $y = x - \sqrt{2}$. These equations determine lines, which in turn bound a diamond shaped region in the plane.

Part A Problem: Show that the unit circle sits inside this diamond tangentially; i.e. show that the unit circle intersects each of the four lines exactly once.

Part A Solution: Graphing the lines:



Solving for the intersection of $x^2 + y^2 = 1$ and $y = x + \sqrt{2}$:

$$x^2 + (x + \sqrt{2})^2 = 1$$
$$2x^2 + 2\sqrt{2}x + 1 = 0$$

The discriminant is $\left(2\sqrt{2}\right)^2 - 4 \cdot 2 \cdot 1 = 0$, so there is only one solution - it intersects the unit circle once. Because $y = x + \sqrt{2}$ intersects the unit circle only once, the unit circle is symmetric around $y = x$, and that $y = x - \sqrt{2}$ is the reflection of $y = x + \sqrt{2}$ about $y = x$, $y = x - \sqrt{2}$ is also tangent to the unit circle.

Because $y = x + \sqrt{2}$ intersects the unit circle only once, the unit circle is symmetric around $x = 0$, and that $y = -x + \sqrt{2}$ is the reflection of $y = x + \sqrt{2}$ about $x = 0$, $y = -x + \sqrt{2}$ is also tangent to the unit circle.

Lastly, using the same logic to determine that $y = x - \sqrt{2}$ is tangent to the unit circle because $y = x + \sqrt{2}$ is tangent previously, we can assert that $y = -x - \sqrt{2}$ is tangent to the unit circle because $y = -x + \sqrt{2}$ is tangent.

Part B Problem: Find the intersection points between the unit circle and each of the four lines.

Part B Solution: Consider the line $y = x + \sqrt{2}$. We found in part A that this line only intersects the unit circle once. If the circle is only to intersect a line A once, then its intersection point can be found by the intersection of a line B that passes through the center of the circle and is perpendicular to A. If B were not perpendicular to A, then the radius would be too long and the circle would intersect A twice.

In this case, line B is $y = -x$. Solving for when lines A and B meet:

$$\begin{aligned} x + \sqrt{2} &= -x \\ 2x &= -\sqrt{2} \\ x &= -\frac{\sqrt{2}}{2} \end{aligned}$$

y can be easily found as $\frac{\sqrt{2}}{2}$. All other lines are simply reflections over the x axis and/or the y-axis of $y = x + \sqrt{2}$. By similarly reflecting the points of intersection, we can find the four intersection points to be:

- $y = x + \sqrt{2}$: $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
- $y = x - \sqrt{2}$: $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
- $y = -x + \sqrt{2}$: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
- $y = -x - \sqrt{2}$: $-\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

Part C Problem: Construct a diamond shaped region in which the circle of radius 1 centered at $(-2, -1)$ sits tangentially. Use the techniques of this section to help.

Part C Solution: The circle's origin is at $(-2, -1)$; it has the same radius as in previous sections, so observations are easily applicable. One thing we can notice is that vertices are placed at $(x_c \pm \frac{\sqrt{2}}{2}, y_c \pm \frac{\sqrt{2}}{2})$ for center (x_c, y_c). Therefore, in this case, two lines should intersect each at $(-2 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2})$, $(-2 - \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2})$, $(-2 + \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2})$, and $(-2 - \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2})$. Two lines should have slope 1, and two should have slope -1. Furthermore, lines of the same slope are $2\sqrt{2}$ in vertical distance from one another.

Finding the line that passes through $\left(-2 + \frac{\sqrt{2}}{2},\ -1 + \frac{\sqrt{2}}{2}\right)$ with slope -1 (we know the line that passes through it is -1 because it is on the top-right side of the unit circle and a line that is tangent to it must have a negative slope):

$$\begin{aligned} -1 + \frac{\sqrt{2}}{2} &= 2 - \frac{\sqrt{2}}{2} + b \\ -3 + \sqrt{2} &= b \end{aligned}$$

$y = -x - 3 + \sqrt{2}$ is the first line; because $(-2 + \sqrt{2}, -1 + \sqrt{2})$ is at the highest point that two lines must intersect at, the other line with slope 1 must be below $y = x + 1$. Using the fact that lines of the same slope are $2\sqrt{2}$ in vertical distance from one another, the other line has equation $y = -x - 3 + \sqrt{2} - 2\sqrt{2} = 3 - \sqrt{2}$.

Finding the line that passes through $\left(-2 - \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}\right)$ with slope 1 (we know the line that passes through it is 1 because it is on the top-left side of the unit circle and a line that is tangent to it must have a positive slope):

$$\begin{aligned} -1 + \frac{\sqrt{2}}{2} &= -2 - \frac{\sqrt{2}}{2} + b \\ 1 + \sqrt{2} &= b \end{aligned}$$

Hence, $y = x + 1 + \sqrt{2}$ is the third line; the fourth line can be derived by translating this line $2\sqrt{2}$ units lower, since $\left(-2 - \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}\right)$ is at the highest point two points can intersect at. The fourth line is hence $y = x + 1 + \sqrt{2} - 2\sqrt{2} = x + 1 - \sqrt{2}$.

The four lines used for constructing the diamond are hence $y = -x - 3 + \sqrt{2}$, $y = -x - 3 - \sqrt{2}$, $y = x + 1 + \sqrt{2}$ and $y = x + 1 - \sqrt{2}$.

Problem 13.2

Problem: The graph of a function $y = f(x)$ is pictured with domain $-2.5 \leq x \leq 3.5$. Sketch the graph of each of the new functions listed: $g(x) = 2f(x+1), h(x) = \frac{1}{2}f(2x-1), j(x) = 4f\left(\frac{1}{3}x + 2\right) - 2$.

Solution:

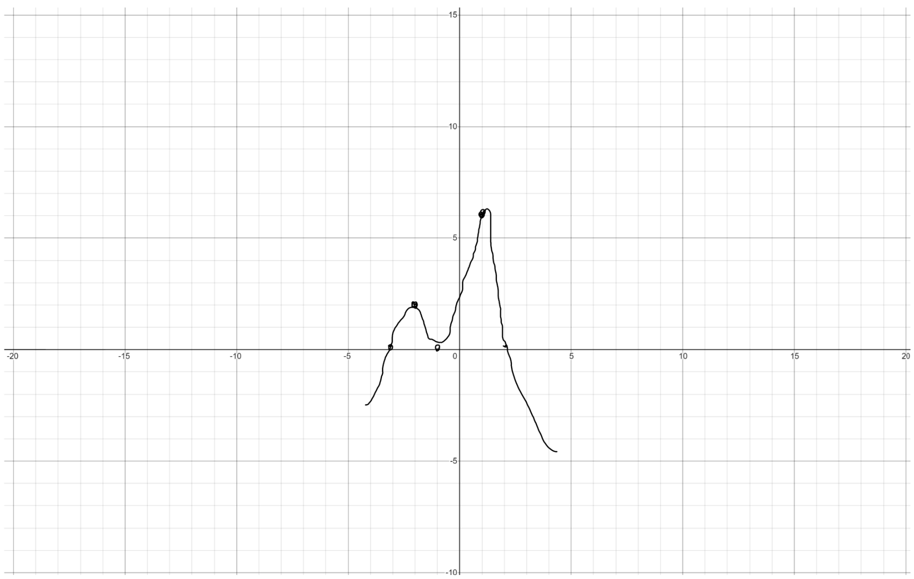


Figure 1: $g(x) = 2f(x+1)$

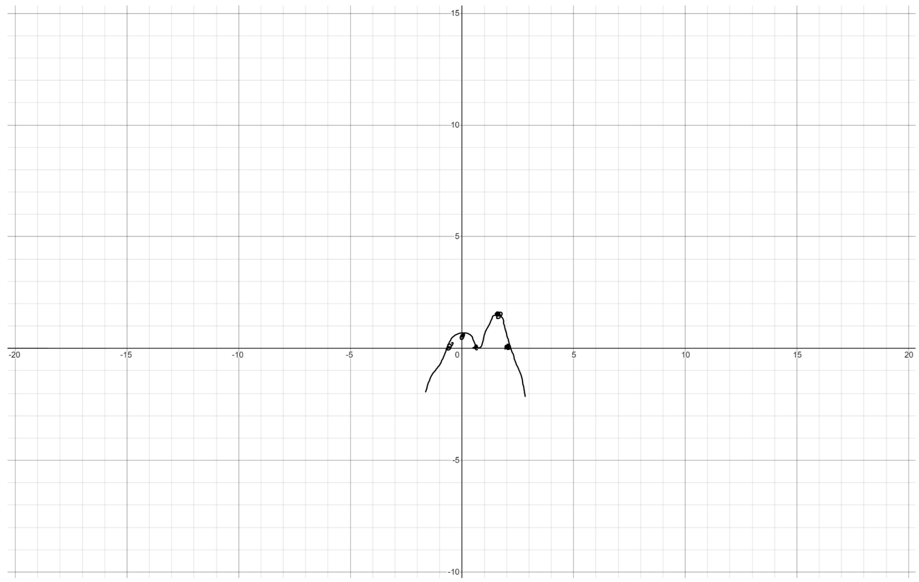


Figure 2: $h(x) = \frac{1}{2}f(2x-1)$

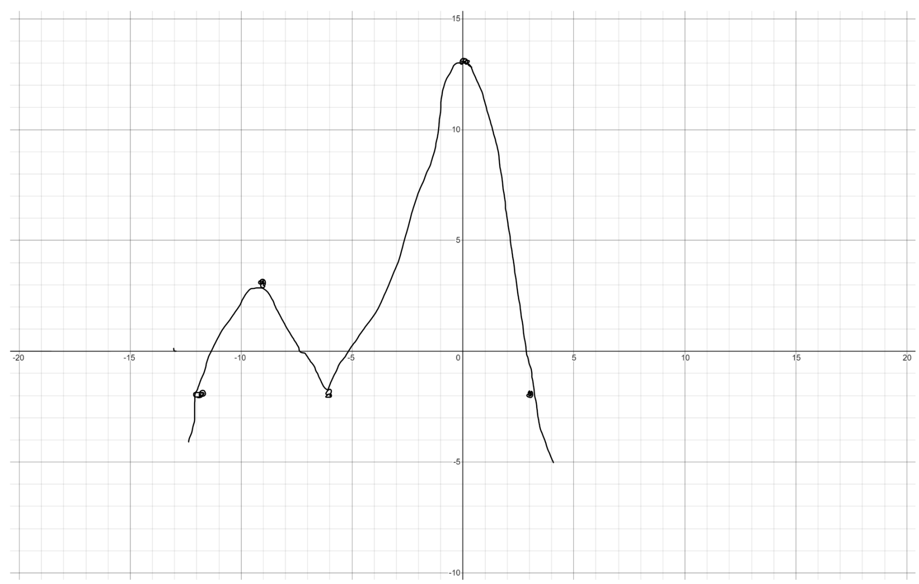


Figure 3: $j(x) = 4f\left(\frac{1}{3}x+2\right)-2$

Problem 13.3

Problem: The graph of a function $y = f(x)$ is pictured with domain $-1 \le x \le 1$. Sketch the graph of the new function $y = g(x) = \frac{1}{\pi}f(3x) - 0.5$. Find the largest possible domain of the function $y = \sqrt{g(x)}$.

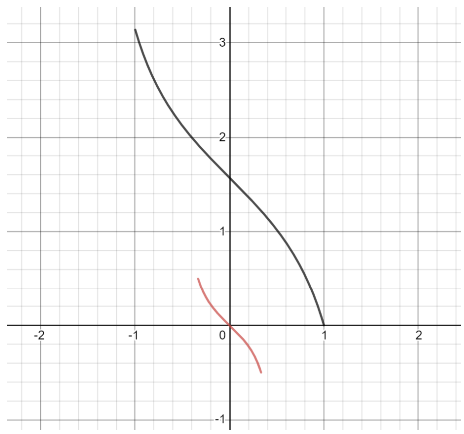
Solution: The graph given can be modelled by $\tilde{f}(x) = \tan(-x) + \frac{\pi}{2}$. We know that in the graph, the curve passes through $(1,0)$; however, our current model predicts $\tilde{f}(1) = \tan(-1) + \frac{\pi}{2} \approx 0.01338\ldots$, which is very close but not at 0. To make a simple fix, we attempt to find the value of a where $\tilde{f}(x) = a\tan(-x) + \frac{\pi}{2}$.

Solving for a:

$$0 = a\tan(-1) + \frac{\pi}{2}$$
$$a\tan(-1) = -\frac{\pi}{2}$$
$$a = \frac{-\frac{\pi}{2}}{\tan(-1)}$$

Hence, our approximate model is $\tilde{f}(x) = \frac{-\frac{\pi}{2}}{\tan(-1)} \cdot \tan(-x) + \frac{\pi}{2}$, with domain restriction $-1 \le x \le 1$.

Plotting this approximation through $g(x) = \frac{1}{\pi} f(3x) - 0.5$ yields the following sketch ($f(x)$ in black, $g(x)$ in red):



We can see that the domain of $g(x)$ is restricted by the domain of $f(x)$. Since $f(x)$ takes in only values of x where $-1 \le x \le 1$, and $g(x)$ passes $3x$ through $f(x)$, $g(x)$'s domain is restricted to $-\frac{1}{3} \le x \le \frac{1}{3}$. Because negative numbers cannot be passed into the real square root function, the domain of $\sqrt{g(x)}$ is $0 \le x \le \frac{1}{3}$.

Alternatively, without plotting, we could determine the new range to be $\left[0, \frac{1}{3}\right]$ because $3x = 0 \to x = 0$ and $3x = 1 \to x = \frac{1}{3}$, forming the new domain for the input of the transformed function.

Problem 13.4

Part A

Part A Context: Each of the six functions $y = f(x)$ can be written in the standard form $y = A|B(x - C)| + D$, for some constants A, B, C, and D. Find these constants, describe the precise order of graphical operations involved in going from the graph of $y = |x|$ to the graph of $y = f(x)$ (paying close attention to the order), write out the multipart rule, sketch the graph, and calculate the coordinates of the vertex of the graph.

Part A1 Problem: $f(x) = |x - 2|$

Part A1 Solution: In this case, only one transformation is being done; thus, it is clear that $A = 1, B = 1, C = 2, D = 0$.

To get from $y = |x|$ to $y = f(x)$, follow the following operations:

1. Shift 2 units to the right.

The x-value of the "vertex" of the graph is the value of x for which the part inside the absolute value is equal to zero. Hence, the vertex is $x - 2 = 0 \to x = 2$. $f(2) = 0$, so the "vertex" occurs at $(2, 0)$.

The multipart rule can be determined as follows:

$$f(x) = \begin{cases} \text{function with positive absolute value} & \text{if } x > x\text{-value of vertex} \\ \text{function with negated absolute value} & \text{if } x \le x\text{-value of vertex} \end{cases}$$

Using this, we can find that the multipart function is

$$f(x) = \begin{cases} x-2 & \text{if } x > 2 \\ -x+2 & \text{if } x \leq 2 \end{cases}$$

Part A2 Problem: $f(x) = 2|x+3|$

Part A2 Solution: This function is already written in $A|B(x-C)|+D$ form; thus, $A = 2, B = 1, C = -3, D = 0$.

To get from $y = |x|$ to $y = f(x)$, follow the following operations:

1. Shift 3 units to the left.
2. Expand vertically by a factor of 2.

The x-value of the "vertex" of the graph is the value of x for which the part inside the absolute value is equal to zero. Hence, the vertex is $x+3 = 0 \rightarrow x = -3$. $f(-3) = 0$, so the "vertex" occurs at $(-3, 0)$.

Using the framework outlined in Part A1, we can find that the multipart function is

$$f(x) = \begin{cases} 2(x+3) & \text{if } x > -3 \\ 2(-(x+3) & \text{if } x \leq -3 \end{cases}$$

Part A3 Problem: $f(x) = |2x-1|$

Part A3 Solution: This function can be written as $f(x) = |2(x-0.5)|$; thus, $A = 1, B = 2, C = 0.5, D = 0$.

To get from $y = |x|$ to $y = f(x)$, follow the following operations:

1. Shift 1 unit to the right.
2. Compress horizontally by a factor of 2.

Alternatively,

1. Compress horizontally by a factor of 2.
2. Shift 0.5 units to the right.

The x-value of the "vertex" of the graph is the value of x for which the part inside the absolute value is equal to zero. Hence, the vertex is $2x-1 = 0 \rightarrow x = \frac{1}{2}$. $f\left(\frac{1}{2}\right) = 0$, so the "vertex" occurs at $\left(\frac{1}{2}, 0\right)$.

Using the framework outlined in Part A1, we can find that the multipart function is

$$f(x) = \begin{cases} 2x-1 & \text{if } x > \frac{1}{2} \\ -(2x-1) & \text{if } x \leq \frac{1}{2} \end{cases}$$

Part A4 Problem: $f(x) = |2(x-1)|$

Part A4 Solution: This function is already in "standard form"; thus, $A = 1, B = 2, C = 1, D = 0$.

To get from $y = |x|$ to $y = f(x) = |2x-2|$, follow the following operations:

1. Shift 1 units to the right.
2. Compress horizontally by a factor of 2.

The x-value of the "vertex" of the graph is the value of x for which the part inside the absolute value is equal to zero. Hence, the vertex is $2x - 2 = 0 \rightarrow x = 1$. $f(1) = 0$, so the "vertex" occurs at $(1, 0)$.

Using the framework outlined in Part A1, we can find that the multipart function is

$$f(x) = \begin{cases} 2(x-1) & \text{if } x > 1 \\ -2(x-1) & \text{if } x \leq 1 \end{cases}$$

Part A5 Problem: $f(x) = 3|2x - 1| + 5$

Part A5 Solution: This function can be written as $3|2(x - 0.5)| + 5$; thus, $A = 3, B = 2, C = 0.5, D = 5$. To get from $y = |x|$ to $y = f(x)$, follow the following operations:

1. Shift 1 units to the right.
2. Compress horizontally by a factor of 2.
3. Expand vertically by a factor of 3.
4. Shift 5 units upwards.

Alternatively,

1. Compress horizontally by a factor of 2.
2. Shift 0.5 units to the right.
3. Expand vertically by a factor of 3.
4. Shift 5 units upwards.

The x-value of the "vertex" of the graph is the value of x for which the part inside the absolute value is equal to zero. Hence, the vertex is $2x - 1 = 0 \rightarrow x = \frac{1}{2}$. $f\left(\frac{1}{2}\right) = 5$, so the "vertex" occurs at $\left(\frac{1}{2}, 5\right)$.

Using the framework outlined in Part A1, we can find that the multipart function is

$$f(x) = \begin{cases} 3(2x-1) + 5 & \text{if } x > \frac{1}{2} \\ -(3(2x-1) + 5) & \text{if } x \leq \frac{1}{2} \end{cases}$$

Part A6 Problem: $f(x) = -2|x + 3| - 1$

Part A3 Solution: This function is already in "standard form"; thus, $A = -2, B = 1, C = -3, D = -1$. To get from $y = |x|$ to $y = f(x)$, follow the following operations:

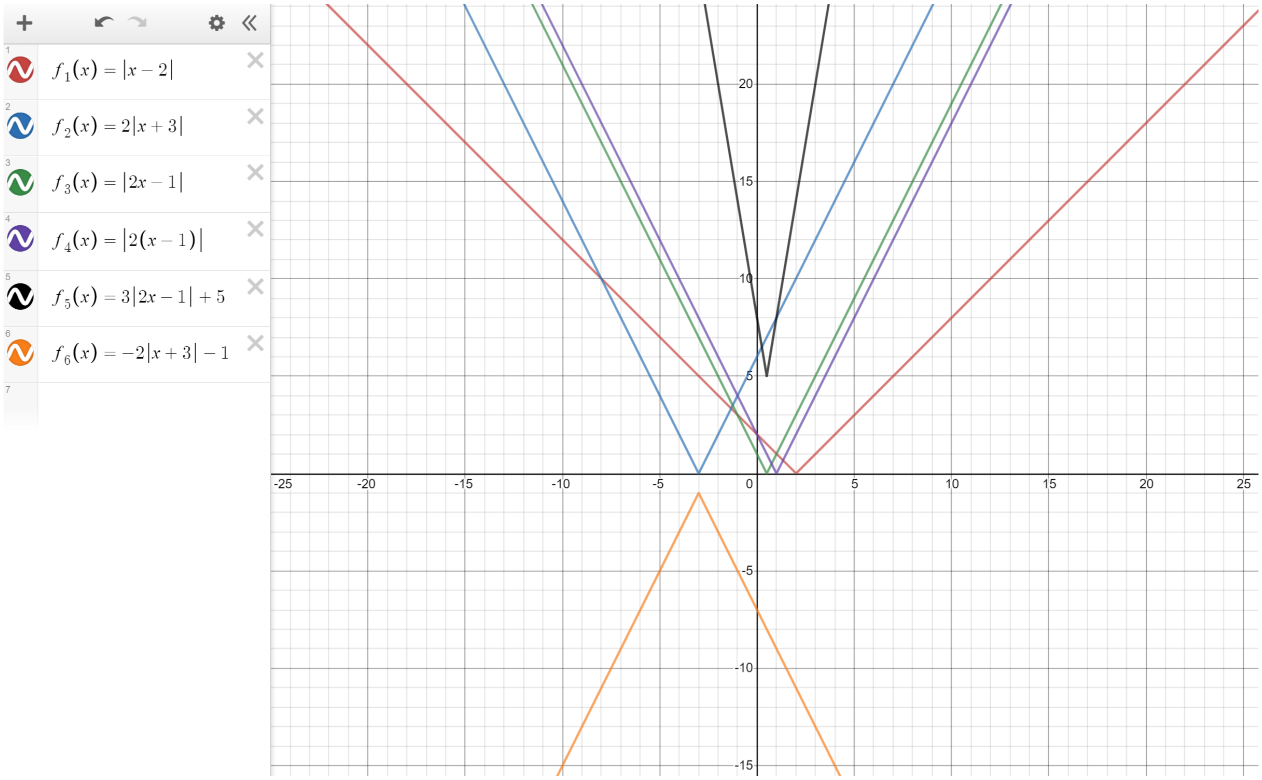
1. Shift 3 units to the left.
2. Expand vertically by a factor of 2.
3. Reflect over the x-axis.
4. Shift 1 unit downwards.

The x-value of the "vertex" of the graph is the value of x for which the part inside the absolute value is equal to zero. Hence, the vertex is $x + 3 = 0 \rightarrow x = -3$. $f(-3) = -1$, so the "vertex" occurs at $(-3, -1)$.

Using the framework outlined in Part A1, we can find that the multipart function is

$$f(x) = \begin{cases} -2(x+3) - 1 & \text{if } x > -3 \\ -(-2(x+3) - 1) & \text{if } x \leq -3 \end{cases}$$

Graphs for Parts A1-A6



Part B

Part B Context: Solve the following inequalities using your work in the previous part of this problem:

Part B1 Problem: $|x - 2| \leq 3$

Part B1 Solution:

$$x - 2 \leq 3 \rightarrow x \leq 5$$

$$-x + 2 \leq 3 \rightarrow -x \leq 1 \rightarrow x \geq -1$$

Hence, $-1 \leq x \leq 5$.

Part B2 Problem: $1 \leq 2|x + 3| \leq 5$

Part B2 Solution:

$$2x + 6 \geq 1 \rightarrow 2x \geq -5 \rightarrow x \geq -\frac{5}{2}$$

$$2x + 6 \leq 5 \rightarrow 2x \leq -1 \rightarrow x \leq -\frac{1}{2}$$

The domain subset $-\frac{5}{2} < x < \frac{1}{2}$ satisfies $1 \leq 2|x + 3| \leq 5$.

$$-2x - 6 \geq 1 \rightarrow -2x \geq 7 \rightarrow x \leq -\frac{7}{2}$$

$$-2x - 6 \leq 5 \rightarrow -2x \leq 11 \rightarrow x \geq -\frac{11}{2}$$

The domain subset $-\frac{11}{2} < x < -\frac{7}{2}$ also satisfies $1 \leq 2|x + 3| \leq 5$.
Hence, the solutions are $-\frac{5}{2} < x < -\frac{1}{2}$ and $-\frac{11}{2} < x < -\frac{7}{2}$.

Part B3 Problem: $y = 3|2x - 1| + 5 \geq 10$

Part B3 Solution:

$$6x - 3 + 5 \geq 10 \rightarrow 6x \geq 8 \rightarrow x \geq \frac{4}{3}$$

$$-6x + 3 + 5 \geq 10 \rightarrow -6x \geq 2 \rightarrow x \leq -\frac{1}{3}$$

Hence, the solution is $x \leq -\frac{1}{3}$ or $x \geq \frac{4}{3}$.

Part C

Part C Problem: The graphs of $y = 3|2x-1| + 5$ and $y = -|x-3| + 10$ intersect to form a bounded region of the plane. Find the vertices of this region and sketch a picture.

Part C Solution: There are four individual line segments:

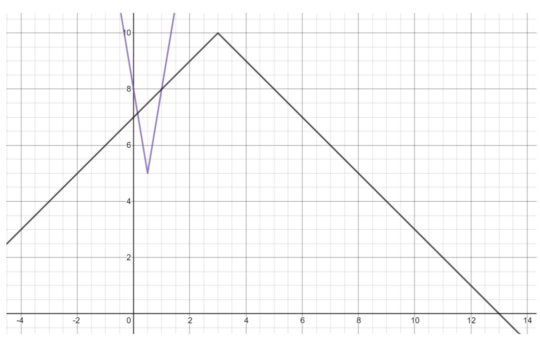
1. $3(2x-1) + 5 \rightarrow 6x - 3 + 5 \rightarrow 6x + 2$, where $2x - 1 > 0 \rightarrow 2x > 1 \rightarrow x > \frac{1}{2}$.
2. $-3(2x-1) + 5 \rightarrow -6x + 3 + 5 \rightarrow -6x + 8$, where $2x - 1 < 0 \rightarrow 2x < 1 \rightarrow x < \frac{1}{2}$.
3. $-(x-3) + 10 \rightarrow -x + 3 + 10 \rightarrow -x + 13$, where $x - 3 > 0 \rightarrow x > 3$.
4. $(x-3) + 10 \rightarrow x + 7$, where $x - 3 < 0 \rightarrow x < 3$.

From the graphs of the equations, we see that Eq. 1 and Eq. 2 intersect with Eq. 4. Thus, we have intersection points to be

- Eq. 1 intersects Eq. 4: $6x + 2 = x + 7 \rightarrow 5x = 5 \rightarrow x = 1$. $y = 1 + 7 = 8$.
- Eq. 2 intersects Eq. 4: $-6x + 8 = x + 7 \rightarrow -7x = -1 \rightarrow x = \frac{1}{7}$. $y = 7 + \frac{1}{7} = \frac{50}{7}$.

Lastly, the third vertex is the "vertex" of $3|2x-1| + 5$, which is $\left(\frac{1}{2}, 5\right)$. Hence, the vertices of this region are $\left(\frac{1}{2}, 5\right), \left(\frac{1}{7}, \frac{50}{7}\right), (1, 8)$.

Graph:



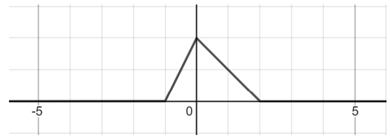
Problem 13.5

Context: Consider the function $y = f(x)$ with multipart definition

$$f(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ 2x + 2 & \text{if } -1 \leq x \leq 0 \\ -x + 2 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{if } x \geq 2 \end{cases}$$

Part A Problem: Sketch the graph of $y = f(x)$.

Part A Solution:

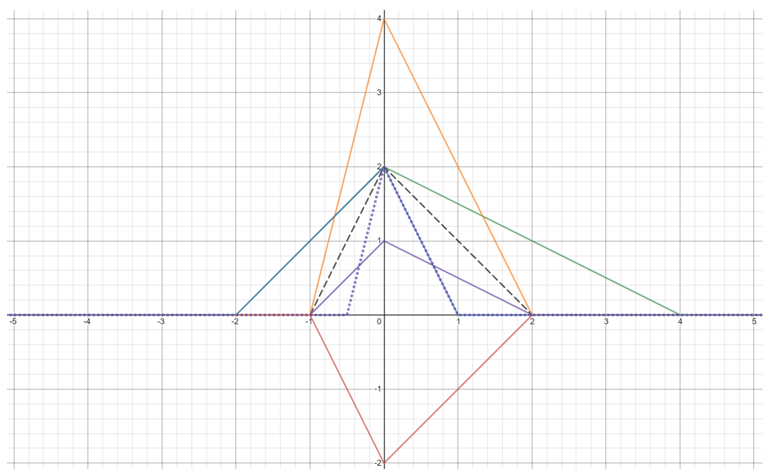


Part B Problem: Is $y = f(x)$ an even function? Is $y = f(x)$ an odd function?

Part B Solution: A function is even if it is symmetric across the y-axis, which is not the case. Hence, $f(x)$ is not even. A function is odd if reflecting it over the x-axis yields the same graph as if it is reflected over the y-axis; this is also not the case. Hence, $f(x)$ is not odd either.

Parts C-E Problems: C) Sketch the reflection of the graph across the x-axis and the y-axis. Obtain the resulting multipart equations for these reflected curves. D) Sketch the vertical dilations $y = 2f(x)$ and $y = \frac{1}{2}f(x)$. E) Sketch the horizontal dilations $y = f(2x)$ and $y = f\left(\frac{1}{2}x\right)$.

Parts C-E Answers: Dotted black is $f(x)$, red is $-f(x)$, full line purple is $\frac{1}{2}f(x)$, green is $f\left(\frac{1}{2}x\right)$, orange is $2f(x)$, blue is $f(-x)$, and dotted purple is $f(2x)$.



The multipart equation for reflection over the x-axis simply requires negating all values; thus it is given by

$$-f(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ -2x - 2 & \text{if } -1 \leq x \leq 0 \\ x - 2 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{if } x \geq 2 \end{cases}$$

The multipart equation for reflecting over the y-axis requires negating values of x to begin with, like this:

$$f(-x) = \begin{cases} 0 & \text{if } -x \leq -1 \\ 2x - 2 & \text{if } -1 \leq -x \leq 0 \\ -x - 2 & \text{if } 0 \leq -x \leq 2 \\ 0 & \text{if } -x \geq 2 \end{cases}$$

We can rewrite the inequality conditions to remove the negation of x, yielding

$$f(-x) = \begin{cases} 0 & \text{if } x \geq 1 \\ 2x - 2 & \text{if } 1 \geq x \geq 0 \\ -x - 2 & \text{if } 0 \geq x \geq -2 \\ 0 & \text{if } x \leq -2 \end{cases}$$

Part F Problem: Find a number $c > 0$ so that the highest point on the graph of the vertical dilation $y = cf(x)$ has y-coordinate 11.

Part F Solution: The maximum point of $f(x)$ is 2; thus we must find the value c such that $2c = 11$, yielding $c = \frac{11}{2}$.

Part G Problem: Using horizontal dilation, find a number $c > 0$ so that the function values $f\left(\frac{x}{c}\right)$ are non-zero for all $-\frac{5}{2} < x < 5$.

Part G Solution: The current nonzero domain of x is $(-1, 2)$; thus must be stretched to $\left(-\frac{5}{2}, 5\right)$. Any number larger than $\frac{5}{2}$ will suffice ($f(x)$ does not necessarily need to be *not* nonzero for values of x outside the domain of $\left(\frac{5}{2}, 5\right)$); for example, $c > \pi^{\pi^{\tau}}$.

Part H Problem: Using horizontal dilation, find positive numbers $c, d > 0$ so that the function values $f\left(\frac{1}{c}(x - d)\right)$ are non-zero precisely when $0 < x < 1$.

Part H Solution: The current nonzero domain of x is $(-1, 2)$; one can first shift this over 1 units yielding a domain of $(0, 3)$. Compressing this by a factor of 3 yields $(0, 1)$. The transformation is hence $f(3x - 1) = f\left(3\left(x - \frac{1}{3}\right)\right)$. Hence, $d = \frac{1}{3}$ and $c = \frac{1}{3}$.

Additional Problem Set Problem 1

Context: Functions can be functions of numbers, but they can also be functions of all sorts of things. In this problem, we'll consider functions of functions, commonly called "functionals".

Part A Problem: Let $\mathcal{F}$ be the function that takes in a function and compresses it by a factor of 2. For example, if $f(x) = x^2$, then $\mathcal{F}(f) = g$, where $g(x) = (2x)^2$. Is $\mathcal{F}$ a one-to-one function? If so, describe its inverse function.

Part A Solution: $\mathcal{F}(g(x)) = g(2x)$. Hence, $\mathcal{F}^{-1}(g(x)) = g(\frac{1}{2}x)$, since

$$\mathcal{F}(\mathcal{F}^{-1}(g(a))) = g\left(2 \cdot \frac{1}{2} \cdot a\right) = g(a)$$

. Because we have found a valid inverse function, $\mathcal{F}$ is one-to-one.

Part B Problem: Let $\mathcal{G}$ be the function that takes in a function and stretches it vertically by a factor of three, and then shifts it to the right by four units. Is $\mathcal{G}$ a one-to-one function? If so, describe its inverse function.

Part B Solution: $\mathcal{G}(g(x)) = 3g(x - 4)$. Hence, $\mathcal{G}^{-1}(g(x)) = \frac{1}{3}g(x + 4)$, since

$$\mathcal{G}(\mathcal{G}^{-1}(g(x))) = 3 \cdot \frac{1}{3} \cdot (g(x + 4 - 4)) = g(x)$$

. Because we have found a valid inverse function, $\mathcal{G}$ is one-to-one.

Part C Problem: Is there a function $\mathcal{H}$ so that $\mathcal{H}$ takes a transformed function and produces the parent function?

Part C Solution: The solution to this answer lies in how broad we are willing to consider the definition of a "function". If it is simply an entity that takes an input and puts out an output, then *yes*, there is a function $\mathcal{H}$ that can take a transformed function and produce the parent function, be that the human mind or a complex mathematical engine. In the scope of functions as simpler transformations, however, no, there is no one function that can through relatively simpler operations return the parent function.

Additional Problem Set 3 Problem 2

Problem: Find a function that satisfies, for all x, $f(x) + 2 = f(x+1)$.

Solution: Looking at the statement in English, it is saying that "moving the function up 2 vertically yields the same result as moving it 1 unit to the left". This sounds like a linear relationship, so let us begin with a linear model $y = mx + b$, where

$$mx + b + 2 = m(x+1) + b \to mx + 2 = mx + m \to m = 2$$

The value of b is arbitrary ($2x + b + 2 = 2(x+1) = b \to b = b$); any value makes it work. So, let the model be

$$f(x) = 2x + {\frac{\operatorname{floor}\left(40 \bmod (9,8)\right)}{4^{45 \bmod \left(\operatorname{ceil}\left(60 \tanh\left(4^{\frac{1}{1+1^{-e}}}\right)\right),\ 3\right)}}}^{\sqrt{(\tan(\sin(5))+100)^{6\frac{\sqrt{\pi+\frac{400}{89}}}{9830989273}}}}$$

We know that our answer is right because a) we found m through rigorously mathematical means, and b) we found that the value of b is arbitrary through rigorously mathematical means.

Additional Problem Set 3 Problem 3

Problem: Find a function which satisfies, for all x, $f(x) + 3 = 4f(x)$.

Solution: We notice in this scenario that unlike in the previous problem, $f(x)$ can be a constant function; the equation resembles a linear equations of sorts, where $f(x)$ represents one variable. Hence, let us take $f(x) = a$ – a constant function – such that the function returns a for any value of x. Then, we have that $a + 3 = 4a \implies 3a = 3 \implies a = 1$. Therefore, a function that satisfies the given equality is $f(x) = 1$. We know this function always satisfies the inequality because $1 + 3$ will always be equal to $4(1)$.

Additional Problem Set 3 Problem 3 Version 2

Problem: Find a function which satisfies, for all x, $f(x+3) = 4f(x)$.

Solution: In English, this reads as "moving the function 3 units to the left yields the same impact as multiplying it by 4". This relationship sounds like an *exponential* one. We can solve for a and b in $f(x) = ab^x$.

$$\begin{aligned} ab^{x+3} &= 4ab^x \\ b^{x+3} &= 4b^x \\ \frac{b^x \cdot b^3}{b^x} &= 4 \\ b^3 &= 4 \\ b &= \sqrt[3]{4} \end{aligned}$$

Given that a cancelled out, it should be clear that a is arbitrary. Hence, our function will be

$$f(x) = \left(\frac{\frac{\operatorname{floor}(40 \bmod (9,8))}{4^{45 \bmod \left(\operatorname{ceil}\left(60 \tanh\left(4^{\frac{1}{1+1^{-e}}}\right)\right),\ 3\right)}}}{\left({\frac{\operatorname{floor}(40 \bmod (9,8))}{4^{45 \bmod \left(\operatorname{ceil}\left(60 \tanh\left(4^{\frac{1}{1+1^{-e}}}\right)\right),\ 3\right)}}}^{\sqrt{(\tan(\sin(5))+100)^{6\frac{\sqrt{\pi+\frac{400}{89}}}{9830989273}}}} \right)^{64\pi^{\tau}}} \right) \cdot \sqrt[3]{4}^{x}$$