Collingwood 48

Andre Ye

8 February 2021

Problem 12.2

Context: As light from the surface penetrates water, its intensity is diminished. In the clear waters of the Caribbean, the intensity is decreased by 15 percent for every 3 meters of depth. Thus, the intensity will have the form of a general exponential function.

Part A Problem: If the intensity of light at the water's surface is $I_\circ$, find a formula for $I(d)$, the intensity of light at a depth of d meters. Your formula should depend on $I_\circ$ and d.

Part A Solution: If the intensity of light decreases by 15 percent for every 3 meters of depth, then 85 percent of the light still remains for every 3 meters of depth. $I_\circ$ represents the initial amount. Hence, the formula is

$$I(d) = I_\circ \cdot 0.85^{\frac{d}{3}}$$

Part B Problem: At what depth will the light intensity be decreased to 1 percent of its surface intensity?

Part B Solution: The formula for percent of light remaining for each meter of depth is $\cdot 0.85^{\frac{d}{3}}$ (the initial amount of light was removed, since percent is not conditional on the initial amount of light). Solving:

$$\begin{aligned} 0.85^{\frac{d}{3}} &= 0.01 \\ \tfrac{d}{3} &= \log_{0.85}(0.01) \\ d &= 3\log_{0.85}(0.01) \approx 85 \text{ meters} \end{aligned}$$

Problem 12.3

Context: Rewrite each function in the form $y = A_\circ \cdot e^{at}$, for appropriate constants $A_\circ$ and a.

Part A Problem: $y = 13(3^t)$

Part A Solution: $y = 13e^{\ln(3)t}$

Part B Problem: $y = 2\left(\frac{1}{8}\right)^t$

Part B Solution: $y = 2e^{\ln\left(\frac{1}{8}\right)t}$

Part C Problem: $y = -7(1.567)^{t-3}$

Part C Solution:

$$-7(1.567)^{t-3} = -7 \cdot \left(\frac{1}{1.567}\right)^3 \cdot (1.567)^t = -7 \cdot \left(\frac{1}{1.567}\right)^3 \cdot (e)^{\ln(1.567)t}$$

Part D Problem: $y = -17(2.005)^{-t}$

Part D Solution:

$$-17(2.005)^{-t} = -17e^{\ln(2.005)\cdot(-t)} = -17e^{-\ln(2.005)t}$$

Part E Problem: $y = 3(14.24)^{4t}$

Part E Solution:

$$3(14.24)^{4t} = 3e^{\ln(14.24)4t} = 3e^{4\ln(14.24)t}$$

Problem 12.4

Part A Problem: If you invest P_0 dollars at 7 percent annual interest and the future value is computed by continuous compounding, how long will it take for your money to double?

Part A Solution: The continuously compounded interest formula is given by $P_0e^{0.07t}$. Solving for when this amount is equal to double the initial amount:

$$\begin{aligned} 2P_0 &= P_0e^{0.07t} \\ e^{0.07t} &= 2 \\ 0.07t &= \ln(2) \\ t &= \frac{\ln(2)}{0.07} \end{aligned}$$

Hence, it will take $\frac{\ln(2)}{0.07}$ years for the money to double.

Part B Problem: Suppose you invest P_0 dollars at r percent annual interest and the future value is computed by continuous compounding. If you want the value of the account to double in 2 years, what is the required interest rate?

Part B Solution: The continuously compounded interest formula is given by P_0e^{rt}. Solving for when this amount is equal to double the initial amount:

$$\begin{aligned} 2P_0 &= P_0e^{rt} \\ e^{rt} &= 2 \\ rt &= \ln(2) \end{aligned}$$

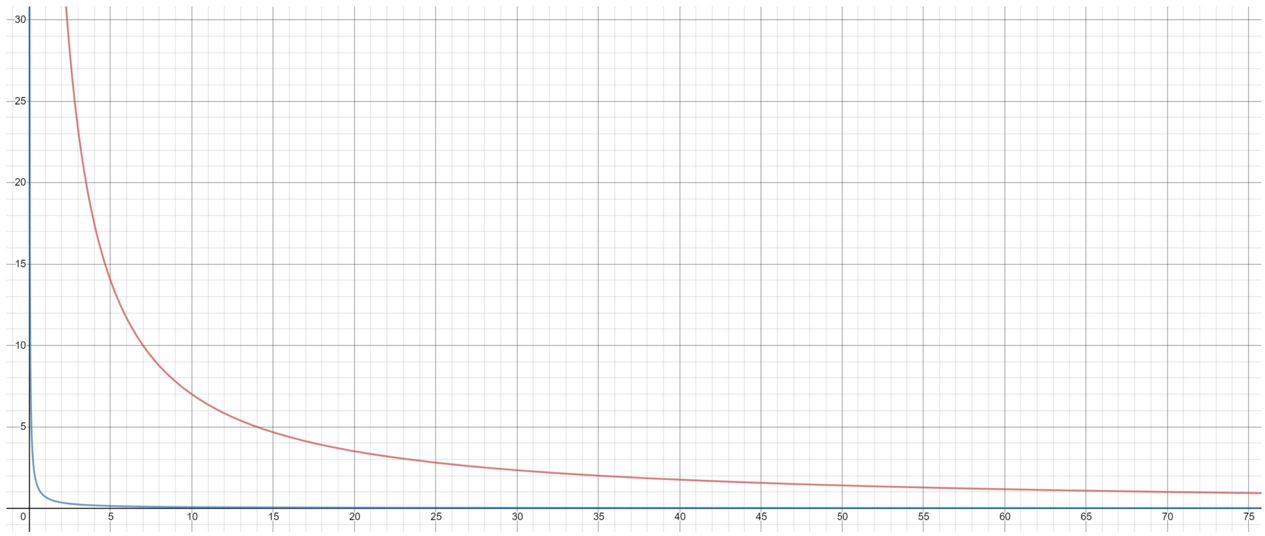
We want the time for this doubling to occur to be two years, so $t = 2$.

$$2r = \ln(2) \implies r = \frac{\ln(2)}{2}$$

Hence, the required interest rate is $\frac{\ln(2)}{2}$ percent.

Part C Problem: A rule of thumb used by many people to determine the length of time to double an investment is the rule of 70. The rule says it takes about $t = \frac{70}{r}$ years to double the investment. Graphically compare this rule to the one isolated in part b. of this problem.

Part C Solution: The rule of thumb states that $t = \frac{70}{r}$; the solution we derived states that $t = \frac{\ln(2)}{r}$. Graphing these out, where the y-axis represents the value of t and the x-axis represents the value of r, yields:



Problem 12.5

Context: The length of some fish are modelled by a *von Bertalanffy* growth function. For Pacific halibut, this function has the form

$$L(t) = 200\left(1 - 0.956e^{-0.18t}\right)$$

where $L(t)$ is the length (in cm) of a fish t years old.

Part A Problem: What is the length of a new-born halibut at birth?

Part A Solution: A new-born halibut at birth should be 0 years old. Using the formula, this gives

$$L(0) = 200\left(1 - 0.956e^{-0.18(0)}\right) = 8.8 \text{ cm}.$$

Part B Problem: Use the formula to estimate the length of a 6-year-old halibut.

Part B Solution:

$$L(6) = 200\left(1 - 0.956e^{-0.18(6)}\right) \approx 135.07 \text{ cm}.$$

Part C Problem: At what age would you expect the halibut to be 120 cm long?

Part C Solution:

$$\begin{aligned} L(t) &= 120 \\ 200\left(1 - 0.956e^{-0.18t}\right) &= 120 \\ 1 - 0.956e^{-0.18t} &= \frac{3}{5} \\ 0.956e^{-0.18t} &= 1 - \frac{3}{5} \\ e^{-0.18t} &= \frac{\frac{2}{5}}{0.956} = \frac{2}{5 \cdot 0.956} \\ -0.18t &= \ln\left(\frac{2}{5 \cdot 0.956}\right) \\ t &= -\frac{\ln\left(\frac{2}{5 \cdot 0.956}\right)}{0.18} \end{aligned}$$

We would expect the halibut to be 120 cm long at $-\frac{\ln(\frac{2}{5 \cdot 0.956})}{0.18}$ years old.

Part D Problem: What is the practical significance of the number 200 in the formula for $L(t)$?

Part D Solution: We can break down the function as follows. Firstly,

$$200\left(1 - 0.956e^{-0.18t}\right) \implies 200\left(1 - 0.956\left(\frac{1}{e}\right)^{0.18t}\right)$$

The inside component is

$$1 - 0.956\left(\frac{1}{e}\right)^{0.18t}.$$

It is $1 -$ an exponential function. Let us analyze the exponential function $0.956\left(\frac{1}{e}\right)^{0.18t}$. Since the base is less than 1, this function's value approaches zero as t approaches infinity. Hence, one minus this function approaches one as t approaches infinity. When multiplied by 200, the function approaches 200 as t approaches infinity. Hence, the practical significance of the number 200 in the formula is that 200 cm is the maximum length of Pacific halibut.

Problem 12.6

Context: A cancerous cell lacks normal biological growth regulation and can divide continuously. Suppose a single mouse skin cell is cancerous and its mitotic cell cycle (the time for the cell to divide once) is 20 hours. The number of cells at time t grows according to an exponential model.

Part A Problem: Find a formula $C(t)$ for the number of cancerous skin cells after t hours.

Part A Solution:

$$C(t) = 2^{\frac{t}{20}}$$

Part B Problem: Assume a typical mouse skin cell is spherical of radius 50×10^{-4} cm. Find the combined volume of all cancerous skin cells after t hours. When will the volume of cancerous cells be 1 cm^3?

Part B Solution: The formula for combined volume of all cancerous skin cells after t hours is $V(t)$:

$$V(t) = 50 \times 10^{-4} \cdot C(t) = 50 \times 10^{-4} \cdot 2^{\frac{t}{20}}$$

Solving for when $V(t) = 1$:

$$\begin{aligned} 2^{\frac{t}{20}} &= \frac{1}{50 \cdot 10^{-4}} \\ \frac{t}{20} &= \log_2\left(\frac{1}{50 \cdot 10^{-4}}\right) \\ t &= 20\log_2\left(\frac{1}{50 \cdot 10^{-4}}\right) \end{aligned}$$

The volume of cancerous cells will be 1 cm^3 after $t = 20\log_2\left(\frac{1}{50 \cdot 10^{-4}}\right)$ hours.

Problem 12.7

Context: Your Grandfather purchased a house for \$55,000 in 1952 and it has increased in value according to a function $y = v(x)$, where x is the number of years owned. These questions probe the future value of the house under various mathematical models.

Part A Question: Suppose the value of the house is \$75,000 in 1962. Assume $v(x)$ is a linear function. Find a formula for $v(x)$. What is the value of the house in 1995? When will the house be valued at \$200,000?

Part A Solution: We have two data points: $(0, 55)$ and $(10, 75)$, where data points are in the form (years since 1952, value in thousands of dollars). The linear model is thus

$$v(x) = \frac{75 - 55}{10}x + 55 = 2x + 55$$

In 1995, $x = 43$; thus the price is estimated to be at $v(43) = 2(43) + 55 = 86 + 55 =$141 thousand dollars.

Solving for when the house will be valued at 200,000 dollars:

$$\begin{aligned} v(x) &= 200 \\ 2x + 55 &= 200 \\ 2x &= 145 \\ x &= 72.5 \end{aligned}$$

The model estimates it will take 72.5 years from 1952 for the house to be valued at 200,000 dollars.

Part B Question: Suppose the value of the house is \$75,000 in 1962 and \$120,000 in 1967. Assume $v(x)$ is a quadratic function. Find a formula for $v(x)$. What is the value of the house in 1995? When will the house be valued at \$200,000?

Part B Solution: We have three data points: $(0, 55)$, $(10, 75)$, and $(15, 120)$, using the same format as in part A. Hence, we have for our quadratic model that:

$$\begin{aligned} 55 &= c \\ 75 &= 100a + 10b + c \\ 120 &= 225a + 15b + c \end{aligned}$$

We have that $20 = 100a + 10b \to 2 = 10a + b$ and $65 = 225a + 15b \to 13 = 45a + 3b$. Substituting $b = 2 - 10a$ into the second equation, we have

$$13 = 45a + 3(2 - 10a) \to 13 = 45a + 6 - 30a \to 7 = 15a \to a = \frac{15}{7}$$

Given that $a = \frac{7}{15}$, we can find b by

$$b = 2 - \frac{70}{15} = -\frac{8}{3}$$

The quadratic model is thus $v(x) = \frac{7}{15}x^2 - \frac{8}{3}x + 55$.

At $x = 43$, in 1995, the house is valued at $v(43) = \frac{7}{15}(43)^2 - \frac{8}{3}(43) + 55 = 803\frac{1}{5}$ thousand dollars.

Solving for when $v(t) = 200$:

$$\begin{aligned} v(x) &= 200 \\ \frac{7}{15}x^2 - \frac{8}{3}x + 55 &= 200 \\ \frac{7x^2}{15} - \frac{8x}{3} - 145 &= 0 \\ \frac{7}{15}x^2 - \frac{40x}{15} - 145 &= 0 \\ 7x^2 - 40x - 2175 &= 0 \end{aligned}$$

$$\begin{aligned} x &= \frac{-(-40) \pm \sqrt{(-40)^2 - 4 \cdot 7(-2175)}}{2 \cdot 7} \\ &= \frac{40 \pm 250}{2 \cdot 7} \\ &= \frac{145}{7}, -15 \end{aligned}$$

Hence, the model predicts it will take $\frac{145}{7}$ years from 1952 for the house to be valued at 200,000 dollars.

Part C Question: Suppose the value of the house is $75,000 in 1962. Assume $v(x)$ is a function of exponential type. Find a formula for $v(x)$. What is the value of the house in 1995? When will the house be valued at $200,000?

Part C Solution: We have two data points: $(0, 55)$ and $(10, 75)$, using the same format as in part A. Hence, we have an exponential model

$$v(x) = a\left(\frac{75}{55}\right)^{\frac{x}{10}}$$

Solving for the parameter a is simple:

$$55 = a\left(\frac{75}{55}\right)^{\frac{0}{10}} \rightarrow a = 55$$

Thus, our exponential model is $v(x) = 55\left(\frac{75}{55}\right)^{\frac{x}{10}}$.

At $x = 43$, in 1995, the house is valued at $v(x) = 55\left(\frac{75}{55}\right)^{\frac{43}{10}} \approx 208.72$ thousand dollars.
Solving for when $v(t) = 200$:

$$\begin{aligned} v(x) &= 200 \\ 55\left(\frac{75}{55}\right)^{\frac{x}{10}} &= 200 \\ \left(\frac{75}{55}\right)^{\frac{x}{10}} &= \frac{200}{55} \\ \frac{x}{10} &= \log_{\frac{75}{55}}(\frac{200}{55}) \\ x &= 10\log_{\frac{75}{55}}\left(\frac{200}{55}\right) \end{aligned}$$

Hence, the model predicts it will take $10\log_{\frac{75}{55}}\left(\frac{200}{55}\right)$ years from 1952 for the house to be valued at 200,000 dollars.

Problem 12.8

Context: Solve the following equations for x.

Part A Problem: $\log_3(5) = \log_2(x)$

Part A Solution:

$$\begin{aligned} \log_3(5) &= \frac{\log_3(x)}{\log_3(2)} \\ \log_3(x) &= \log_3(5) \cdot \log_3(2) \\ x &= 3^{\log_3(5)\cdot\log_3(2)} \end{aligned}$$

Part B Problem: $10^{\log_2(x)} = 3$

Part B Solution:

$$\begin{aligned} 10^{\log_2(x)} &= 3 \\ \log_2(x) &= \log_1 0(3) \\ \log_2(x) &= \frac{\log_2(3)}{\log_2(10)} \\ x &= 2^{\frac{\log_2(3)}{\log_2(10)}} \end{aligned}$$

Part C Problem: $3^{5^x} = 7$

Part C Solution:

$$\begin{aligned} 5^x &= \log_3(7) \\ x &= \log_5(\log_3(7)) \end{aligned}$$

Part D Problem: $\log_2(\ln(x)) = 3$

Part D Solution:

$$\begin{aligned} 3 &= \log_2(\ln(x)) \\ 2^3 &= \ln(x) \\ x &= e^8 \end{aligned}$$

Part E Problem: $e^x = 10^5$

Part E Solution:

$$x = \ln\left(10^5\right)$$

Part F Problem: $2^{3x+5} = 3^2$

Part F Solution:

$$\begin{aligned} 2^{3x+5} &= 3^2 \\ 3x + 5 &= \log_2\left(3^2\right) \\ x &= \frac{\log_2\left(3^2\right) - 5}{3} \end{aligned}$$

Problem 12.9

Context: A ship embarked on a long voyage. At the start of the voyage, there were 500 ants in the cargo hold of the ship. One week into the voyage, there were 800 ants. Suppose the population of ants is an exponential function of time.

Part A Problem: How long did it take the population to double?

Part A Solution: Let us first construct an exponential model. We have two data points, (0, 5) and (1, 8), in the form (weeks, hundreds of ants). Our model is thus

$$5\left(\frac{8}{5}\right)^t$$

Solving for when this equals $5r$, or r times the initial population.

$$5r = 5\left(\frac{8}{5}\right)^t$$
$$\left(\frac{8}{5}\right)^t = r$$
$$t = \log_{\frac{8}{5}}(r)$$

In this case of doubling, $r = 2$. Hence, it took $\log_{\frac{8}{5}}(2)$ weeks for the population to double.

Part B Problem: How long did it take the population to triple?

Part B Solution: Using the previous formula, tripling means $r = 3$; thus, it took $\log_{\frac{8}{5}}(3)$ weeks for the population to triple.

Part C Problem: When were there be 10,000 ants on board?

Part C Solution: Let us find r when there are 10,000 ants; in this case, there are 100 hundreds of ants, and this means that $r = \frac{100}{5} = 20$. Using the formula, it took $\log_{\frac{8}{5}}(20)$ weeks for the population to reach 10,000 ants.

Part D Problem: There also was an exponentially-growing population of anteaters on board. At the start of the voyage there were 17 anteaters, and the population of anteaters doubled every 2.8 weeks. How long into the voyage were there 200 ants per anteater?

Part D Solution: The formula for number of hundreds of anteaters can be modelled by $e(t) = \frac{17}{100}(2)^{\frac{t}{2.8}}$. Given that the number of hundreds of ants can be given by $a(t) = 5\left(\frac{8}{5}\right)^t$, we want to solve for when $\frac{a(t)}{e(t)} = 200$.

$$\frac{5\left(\frac{8}{5}\right)^x}{\frac{17}{100}(2)^{\frac{x}{2.8}}} = 200$$
$$5\left(\frac{8}{5}\right)^x = 200\left(\frac{17}{100}(2)^{\frac{x}{2.8}}\right)$$
$$5\left(\frac{8}{5}\right)^x = 34\left(\sqrt[2.8]{2}\right)^x$$
$$\frac{\left(\frac{8}{5}\right)^x}{\left(\sqrt[2.8]{2}\right)^x} = \frac{34}{5}$$
$$\left(\frac{\frac{8}{5}}{\sqrt[2.8]{2}}\right)^x = \frac{34}{5}$$
$$x = \log_{\frac{\frac{8}{5}}{\sqrt[2.8]{2}}}\left(\frac{34}{5}\right)$$

Hence, it will take $\log_{\frac{\frac{8}{5}}{\sqrt[2.8]{2}}}\left(\frac{34}{5}\right)$ weeks for the ratio of 200 ants per anteater to be met.

Problem 12.10

Solution: The populations of termites and spiders in a certain house are growing exponentially. The house contains 100 termites the day you move in. After 4 days, the house contains 200 termites. Three days after moving in, there are two times as many termites as spiders. Eight days after moving in, there were four times as many termites as spiders. How long (in days) does it take the population of spiders to triple?

Solution: Let us first construct a model for the number of termites after x days; the problem gives two data points, (0, 100) and (4, 200), in the form (days, number of termites). The appropriate exponential model is hence $100(2)^{\frac{x}{4}}$ This predicts that after three days, there will be $100(2)^{\frac{3}{4}} = 168.17928\ldots$; after eight days, the model predicts there will be $100(2)^{\frac{8}{4}} = 400$.

This gives us two data points – using that the number of spiders is $\frac{1}{2}$ and $\frac{1}{4}$ of the number of termites, respectively – to model the number of spiders after x days: $(3, \frac{168.17928}{2})$ and $(8, 100)$. This gives us a model $a\left(\frac{100}{\frac{168.17928}{2}}\right)^{\frac{x}{5}}$. To find the value of a, we can solve using $(8, 400)$:

$$400 = a\left(\frac{100}{\frac{168.17928}{2}}\right)^{\frac{8}{5}}$$

$$a = \frac{400}{\left(\frac{100}{\frac{168.17928}{2}}\right)^{\frac{8}{5}}}$$

Hence, the model for the spider population is $\frac{400}{\left(\frac{100}{\frac{168.17928}{2}}\right)^{\frac{8}{5}}}\left(\frac{400}{168.17928}\right)^{\frac{x}{5}}$. To solve for when the population triples, we can ignore the coefficient – lamenting that we spent quite a bit of effort finding it without the foresight to realize it would not be needed – and consider when the exponential aspect equals 3.

$$\left(\frac{100}{\frac{168.17928}{2}}\right)^{\frac{8}{5}} = 3$$

$$\sqrt[5]{\frac{100}{\frac{168.17928}{2}}}^{x} = 3$$

$$x = \log_{\sqrt[5]{\frac{100}{\frac{168.17928}{2}}}}(3)$$

Therefore, it takes $\log_{\sqrt[5]{\frac{100}{\frac{168.17928}{2}}}}(3)$ days for the spider population to triple.

Problem 12.11

Context: In 1987, the population of Mexico was estimated at 82 million people, with an annual growth rate of 2.5 percent. The 1987 population of the United States was estimated at 244 million with an annual growth rate of 0.7 percent. Assume that both populations are growing exponentially.

Part A Problem: When will Mexico double its 1987 population?

Part A Solution: Mexico's population in millions can be modelled by the equation $82 \cdot 1.025^x$, where t is the number of years since 1987. Solving for when the rate equals two:

$$1.025^x = 2$$

$$t = \log_{1.025}(2)$$

Mexico will double its population $\log_{2.025}(2)$ years from 1987.

Part B Problem: When will the United States and Mexico have the same population?

Part B Solution: The equation for America's population is $244(1.007)^t$. Solving for when these are equal:

$$244(1.007)^t = 82 * (1.025)^t$$

$$\left(\frac{1.007}{1.025}\right)^t = \frac{82}{244}$$

$$t = \log_{\frac{1.007}{1.025}}\left(\frac{82}{244}\right)$$

Hence, the United States and Mexico will have the same population $\log_{\frac{1.007}{1.025}}\left(\frac{82}{244}\right)$ years from 1987.

Problem 12.12

Context: The cities of Abnarca and Bonipto have populations that are growing exponentially. In 1980, Abnarca had a population of 25,000 people. In 1990, its population was 29,000. Bonipto had a population of 34,000 in 1980. The population of Bonipto doubles every 55 years.

Part A Problem: How long does it take the population of Abnarca to double?

Part A Solution: Abnarca's population can be modelled by $a(t) = 25\left(\frac{29}{25}\right)^{\frac{t}{10}}$, where $a(t)$ takes in t – the number of years since 1980 – and outputs the population of people in thousands. Considering only the exponential components, we have that

$$\begin{aligned}\left(\frac{29}{25}\right)^{\frac{t}{10}} &= 2\\ \frac{t}{10} &= \log_{\frac{29}{25}}(2)\\ t &= 10\log_{\frac{29}{25}}(2)\end{aligned}$$

The population of Abnarca will double $10\log_{\frac{29}{25}}(2)$ years after 1980.

Part B Problem: When will Abnarca's population equal that of Bonipto?

Part B Solution: Bonipto's population can be modelled by $b(t) = 34(2)^{\frac{t}{55}}$, where $b(t)$ takes in t – the number of years since 1980 – and outputs the number of people in thousands. By setting $a(t) = b(t)$, we can solve for when the two populations will be equal.

$$\begin{aligned}a(t) &= b(t)\\ 25\left(\frac{29}{25}\right)^{\frac{t}{10}} &= 34(2)^{\frac{t}{55}}\\ 25\left(\sqrt[10]{\frac{29}{25}}\right)^{t} &= 34\left(\sqrt[55]{2}\right)^{t}\\ \left(\frac{\sqrt[10]{\frac{29}{25}}}{\sqrt[55]{2}}\right)^{t} &= \frac{34}{25}\\ t &= \log_{\frac{\sqrt[10]{\frac{29}{25}}}{\sqrt[55]{2}}}\left(\frac{34}{25}\right)\end{aligned}$$

The two populations will then be equal $\log_{\frac{\sqrt[10]{\frac{29}{25}}}{\sqrt[55]{2}}}\left(\frac{34}{25}\right)$ years after 1980.