Collingwood 46

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1 Question 11.1

11.1 Context: In 1968, the U.S. minimum wage was \$1.60 per hour. In 1976, the minimum wage was \$2.30 per hour. Assume the minimum wage grows according to an exponential model $w(t)$, where t represents the time in years after 1960.

11.1a Problem: Find a formula for $w(t)$.

11.1a Solution: We have two points: $(8, 1.6)$ and $(16, 2.3)$. An exponential function takes the form $y = ab^x$. Thus, we have that $1.6 = ab^8$ and $2.3 = ab^{16}$. Solving:

$$1.6 = ab^8 \implies a = \frac{1.6}{b^8}$$
$$2.3 = \left(\frac{1.6}{b^8}\right) b^{16}$$
$$2.3 = 1.6b^8$$
$$b^8 = \frac{2.3}{1.6}$$
$$b = \sqrt[8]{\frac{2.3}{1.6}} \approx 1.046408$$
$$1.6 = a\left(\sqrt[8]{\frac{2.3}{1.6}}\right)^8$$
$$1.6 = \frac{2.3}{1.6}a$$
$$a = 1.6 \cdot \frac{1.6}{2.3} \approx 1.11304$$

Thus, the exponential model is about $1.11304(1.046408)^x$.

11.1b Problem: What does the model predict for the minimum wage in 1960?

11.1b Solution: Plugging in $x = 0$ yields $1.113 \cdot 1.0464 = 1.164694 \approx \$1.16$.

11.1c Problem: If the minimum wage was \$5.15 in 1996, is this above, below or equal to what the model predicts?

11.1c Solution: Our model calculates the minimum wage to be $1.11304(1.046408)^{36} \approx 5.698343 \approx \$5.70$ by 1996. Therefore, the minimum wage at 1996 is below what the model predicts.

2 Question 11.2

11.2 Context: The town of Pinedale, Wyoming, is experiencing a population boom. In 1990, the population was 860 and five years later it was 1210.

11.2a Problem: Find a linear model $l(x)$ and an exponential model $p(x)$ for the population of Pinedale in the year $1990 + x$.

11.2a Solution: We have two data points: $(0, 860)$ and $(5, 1210)$.

To build a linear model, we find the slope, which is $\frac{1210-860}{5} = 70$. The linear model is thus $l(x) = 70x + 860$.

To build an exponential model, we can solve for a and b in $p(x) = ab^x$. We have that $860 = ab^0$ and $1210 = ab^5$. From the former equation, it is true that $a = 860$. Using this to find b:

$$1210 = 860b^5$$
$$\frac{1210}{860} = b^5$$
$$b = \sqrt[5]{\left(\frac{1210}{860}\right)} \approx 1.070674$$

Hence, the exponential model is (approximately) $p(x) = 860(1.070674)^x$

11.2b Problem: What do these models estimate the population of Pinedale to be in the year 2000?

11.2b Solution: $x = 10$ at the year 2000; thus we our estimates for the models are

$$l(10) = 70(10) + 860 = 1560 \text{ people}$$

$$p(10) = 860(1.070674)^{10} \approx 1702.43686 \text{ people}$$

3 Question 11.3

11.3 Problem: In 1989, research scientists published a model for predicting the cumulative number of AIDS cases reported in the United States:

$$a(t) = 155\left(\frac{t - 1980}{10}\right)^3, \text{ thousands}$$

where t is the year. This paper was considered a "relief", since there was a fear the correct model would be of exponential type. Pick two data points predicted by the research model $a(t)$ to construct a new exponential model $b(t)$ for the number of cumulative AIDS cases. Discuss how the two models differ and explain the use of the word "relief"".

11.3 Solution: Let us take two data points: $t = 1990$ and $t = 2000$. These yield the data points:

$$a(1980) = 155\left(\frac{1990 - 1980}{10}\right)^3 = 155 \implies (1990, 155)$$

$$a(1990) = 155\left(\frac{2000 - 1980}{10}\right)^3 = 310 \implies (2000, 310)$$

. Thus, using the model $b(t) = jk^t$, we have:

$$155 = jk^{1990}$$

$$310 = jk^{2000}$$

We can use these statements to solve for j and k:

$$
\begin{aligned}
155 &= jk^{1990} \\
j &= \frac{155}{k^{1990}} \\
\left(\frac{155}{k^{1990}}\right) k^{2000} &= 310 \\
155k^{10} &= 310 \\
k^{10} &= \frac{310}{155} \\
k &= \sqrt[10]{2} \\
j &= \frac{155}{\left(\sqrt[10]{2}\right)^{1990}} \\
j &= \frac{155}{2^{199}}
\end{aligned}
$$

The exponential model is therefore $b(t) = \frac{155}{2^{199}} \left(\sqrt[10]{2}\right)^{x}$. As a comparison, consider the following data table:

t	$a(t)$	$b(t)$
2000	1240	310
2050	53,165	9,920
2100	267,840	317,440
2150	761,515	10,158,080
2200	1,650,440	325,058,560

We assume "relief" in this context means that as time progresses, a cubic function will inevitably succumb to the incredible compounding power of exponents. Fast-forward 180 years, and the exponential model predicts approximately 196 times more cumulative AIDS cases than the cubic one.

4 Question 4

11.4 Context: Define two new functions:

$$y = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

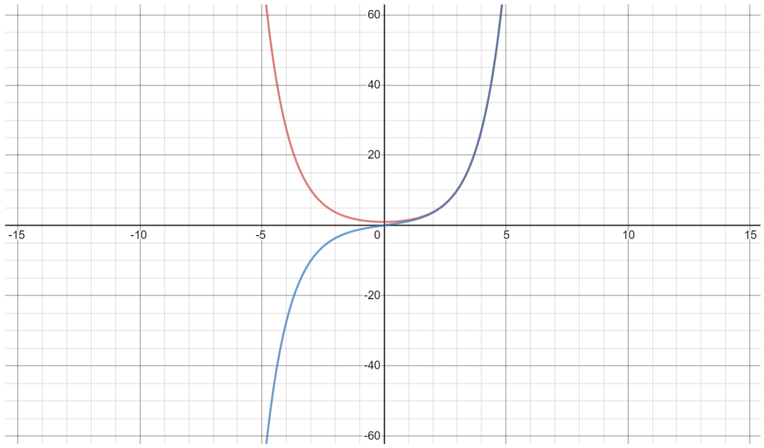
and

$$y = \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

These are called the basic *hyperbolic trigonometric functions.*

11.4a Problem: Sketch rough graphs of these two functions.

11.4a Solution: cosh in red, sinh in blue.



11.4b Problem: The graph of the equation $x^2 - y^2 = 1$ is shown below; this is called the *unit hyperbola*. For any value a, show that the point $(x, y) = (\cosh(a), \sinh(a))$ is on the unit hyperbola.

11.4b Solution: Substituting x and y for $\cosh(a)$ and $\sinh(a)$:

$$
\begin{aligned}
(\cosh(a))^2 - (\sinh(a))^2 &= 1 \\
\left(\frac{e^a + e^{-a}}{2}\right)^2 - \left(\frac{e^a - e^{-a}}{2}\right) &= 1 \\
\frac{(e^a + e^{-a})^2}{4} - \frac{(e^a - e^{-a})^2}{4} &= 1 \\
\left(e^a + e^{-a}\right)^2 - \left(e^a - e^{-a}\right)^2 &= 4 \\
(e^{a^2} + 2e^a e^{-a} + e^{(-a)^2}) - (e^{a^2} - 2e^a e^{-a} + e^{(-a)^2}) &= 4 \\
(e^{a^2} + 2 + e^{a^2}) - (e^{a^2} - 2 + e^{a^2}) &= 4 \\
e^{a^2} + 2 + e^{a^2} - e^{a^2} + 2 - e^{a^2} &= 4 \\
4 &= 4
\end{aligned}
$$

Since $4 = 4$, we have shown that $(\cosh(a))^2 - (\sinh(a))^2 = 1$.

11.4c Problem: The hanging cable is modelled by a portion of the graph of the function

$$y = a\text{cosh}\left(\frac{x-h}{a}\right) + C,$$

for appropriate constants a, h and C. The constant h depends on how the coordinate system is imposed. A cable for a suspension bridge hangs from two 100 ft. high towers located 400 ft. apart. Impose a coordinate system so that the picture is symmetric about the y-axis and the roadway coincides with the x-axis. The hanging cable constant is $a = 500$ and $h = 0$. Find the minimum distance from the cable to the road.

11.4c Solution: Using $a = 500$ and $h = 0$, the equation modelling a portion of the hanging cable is

$$y = 500\text{cosh}\left(\frac{x}{500}\right) + C.$$

Graphing the hanging cable reveals that h represents how far left or right the expression is shifted; that $h = 0$ means in the coordinate system the cable is symmetrical about $x = 0$, and the value of $x = 0$ is the lowest value of y. Hence, plugging in $x = 0$:

$$\begin{aligned}
&500\cosh\left(\frac{0}{500}\right)+C\\
&=500\cosh(0)+C\\
&=500\frac{e^0+e^{-0}}{2}+C\\
&=500+C
\end{aligned}$$

Therefore, the lowest point is $(0, 500+c)$. Thus, the minimum distance from the cable to the road is $500+C$. To find the value of C, we can find the equation that goes through the point $(200, 100)$.

$$\begin{aligned}
100 &= 500\cosh\left(\frac{200}{500}\right)+C\\
100 &= 500\left(\frac{e^{\left(\frac{2}{5}\right)}+e^{-\left(\frac{2}{5}\right)}}{2}\right)+C\\
C &= 100-500\left(\frac{e^{\left(\frac{2}{5}\right)}+e^{-\left(\frac{2}{5}\right)}}{2}\right)\\
&= -440.536185919
\end{aligned}$$

As we have derived, the lowest point is $\approx 500-440.536185919 \approx 59.463814081$ feet.

5 Additional Problem Set Question 1

Problem: Uranium-235 has a half-life of about 700 million years. A group of dinosaur scientists buried one metric ton (one thousand kilograms) of uranium-235 deep in the Nevada desert, and then promptly went extinct. Write a formula for the amount of uranium-235 remaining after t millions of years, and use it to determine the amount of uranium-235 remaining now, after 65 million years.

Solution: It takes uranium-235 700 millions of years to decrease by half. The initial amount of uranium-235 is 1 metric ton. Therefore, the amount remaining after t years is

$$1\left(\frac{1}{2}\right)^{\frac{t}{700}} \text{ metric tons}$$

This means that after 65 million years, there amount of uranium-235 remaining is

$$1\left(\frac{1}{2}\right)^{\frac{65}{700}} \approx 0.93766 \text{ metric tons.}$$

6 Additional Problem Set Question 2

Problem: A job advertisement promises a 3% raise every year and a starting salary of $50,000. Write a formula for the job's predicted salary after t years, and use it to predict the salary of an extremely senior employee who's worked there for 100 years.

Solution: If there is a 3 percent raise every year, then it is multiplied by 1.03 every year. Therefore, the model is

$$50000(1.03)^t$$

The salary of a senior employee who has worked there for 100 years is

$$50000(1.03)^t = 50000(1.03)^{100} \approx \$960,931.60\ldots \text{ per year}$$

7 Additional Problem Set Question 3

Problem: An ecologist notices that the population of invasive butterflies tripled in the last five years. Assuming that this trend continues, and that the current population is 1000, how many invasive butterflies will there be after t years?

Solution: The model is

$$1000(3)^{\frac{t}{5}} \text{ invasive butterflies}.$$

8 Additional Problem Set Question 4

Context: Another ecologist is studying the populations of two species of deer in a particular region. She notices that the population of one species is doubling every eight years, and the population of the other species is tripling every twelve years. The current populations are 500 and 900 respectively.

4a Problem: Which population is growing more quickly?

4a Solution: The models for the species of deer are:

1. $500 \cdot (2)^{\frac{t}{8}}$
2. $900 \cdot (3)^{\frac{t}{12}}$

The questions asks which population grows more quickly, so the initial population is not relevant. Thus, the important components of each model that indicate *growth* are the exponentials. Converting these exponentials to be in the form b^t:

$$2^{\frac{t}{8}} = \sqrt[8]{2}^t \approx = 1.090508^t$$
$$3^{\frac{t}{12}} = \sqrt[12]{3}^t \approx 1.095873^t$$

A higher base means that the population grows faster. Therefore, the second species of deer grows faster.

4b Problem: Write a formula for the total number of deer in the region, counting both species, after t years.

4b Solution: Adding the populations from the two models together yields $p(t) = 500 \cdot (2)^{\frac{t}{8}} + 900 \cdot (3)^{\frac{t}{12}}$.

4c Problem: Is the function you wrote in (b) exponential?

4c Solution: No, because it cannot be written in the exponential form ab^x. In order to combine two exponentials through addition like $a_1 b_1^x + a_2 b_2^x$, it must be true that $b_1 = b_2$, and the resulting expression is $(a_1 + a_2)b_1^x$. As was shown through simplification in 4a, this is not true; therefore, the function cannot be put into exponential form.