Collingwood 46

Andre Ye

25 January 2021

1 Question 10.2

10.2 Context: Put each equation in standard exponential form.

Part	Problem	Solution
a	$y = 3\left(2^{-x}\right)$	$3\left(2^{-x}\right) = 3\left(\left(2^{-1}\right)^{x}\right) = 3\left(\frac{1}{2}\right)^{x}$
b	$y = 4^{-\frac{x}{2}}$	$4^{-\frac{x}{2}} = 4^{\left(\frac{1}{2}\right)^{(-x)}} = \sqrt{4}^{(-1)^{x}} = \left(\frac{1}{2}\right)^{x}$
c	$y = \pi^{\pi x}$	$y = \left(\pi^{\pi}\right)^{x}$ (π^{π} is the base)
d	$y = 1\left(\frac{1}{3}\right)^{3+\frac{x}{2}}$	$1\left(\frac{1}{3}\right)^{3+\frac{x}{2}} = \left(\frac{1}{3}\right)^{3} \cdot \left(\frac{1}{3}\right)^{\frac{x}{2}} = \frac{1}{27} \cdot \left(\frac{1}{3}\right)^{\left(\frac{1}{2}\right)^{x}} = \frac{1}{27} \cdot \left(\frac{1}{\sqrt{3}}\right)^{x}$
e	$y = \frac{5}{0.345^{2x-7}}$	$\frac{5}{0.345^{2x-7}} = \frac{5}{\frac{0.345^{2x}}{0.345^{7}}} = \frac{5 \cdot 0.345^{7}}{0.345^{2x}} = 5 \cdot 0.345^{(7-2x)}$ $= 5 \cdot 0.345^{7} \cdot \left(\frac{1}{0.345^{2}}\right)^{x}$

2 Question 10.3

10.3 Context: A colony of yeast cells is estimated to contain 106 cells at time $t = 0$. After collecting experimental data in the lab, you decide that the total population of cells at time t hours is given by the function

$$y = 10^{6} e^{0.495105t}$$

10.3a Problem: How many cells are present after one hour?

10.3a Solution: After one hour, there are

$$y = 10^{6} \cdot e^{0.495105(1)} \approx 1.641 \times 10^{6} \text{ cells}$$

10.3b Problem: (True or False) The population of yeast cells will double every 1.4 hours.

10.3b Solution: Let us find the rate of change at $t = 0$ and $t = 1.4$:

$$\frac{e^{0.495105(1.4)}}{e^{0.495105(0)}} = 1.99999 \cdots = 2$$

Thus, the answer is True, the population doubles every 1.4 hours. We only need to test one pair of points because this function is exponential, meaning calculating the "rate" in this fashion for some arbitrary value of a, $\frac{e^{0.495105(a+1.4)}}{e^{0.495105(a)}}$, is the same everywhere.

10.3c Problem: Cherie, another member of your lab, looks at your notebook and says : ...that formula is wrong, my calculations predict the formula for the number of yeast cells is given by the function

$$y = 10^6(2.042727)^{0.693147t}$$

Should you be worried by Cherie's remark?

10.3c Solution: Simplifying Cherie's equation:

$$\begin{aligned} y &= 10^6(2.042727)^{0.693147t} \\ &= 10^6\left(2.042727^{0.693147}\right)^t \\ &= 10^6(1.64067)^t \end{aligned}$$

Simplifying our equation:

$$\begin{aligned} y &= 10^6 e^{0.495105t} \\ y &= 10^6\left(e^0.495105\right)^t \\ y &= 10^6(1.64067)^t \end{aligned}$$

These two are identical when simplified. Thus, besides Cherie's odd choice of a base and the fact that e is many times prettier than 2.042727, we don't need to worry about the mathematical integrity of the equation.

10.3d Problem: Anja, a third member of your lab working with the same yeast cells, took these two measurements: 7.246×10^6 cells after 4 hours; 16.504×10^6 cells after 6 hours. Should you be worried by Anja's results? If Anja's measurements are correct, does your model over estimate or under estimate the number of yeast cells at time t?

10.3d Solution: Using our equation:

$$10^6 e^{0.495105(4)} \approx 10^6 \cdot 7.24578$$

$$10^6 e^{0.495105(6)} \approx 10^6 \cdot 19.50420$$

Our model is the same as Anja's for $t = 4$, but larger than Anja's estimate for $t = 6$. This discrepancy is concerning, so either our measurements are wrong, or Anja's measurements are wrong. Thus, if Anja's observations are correct, our model overestimates the number of yeast cells.

3 Question 10.5

10.5 Context: You have a chess board as pictured, with squares numbered 1 through 64. You also have a huge change jar with an unlimited number of dimes. On the first square you place one dime. On the second square you stack 2 dimes. Then you continue, always doubling the number from the previous square.

10.5a Problem: How many dimes will you have stacked on the 10th square?

10.5a Solution: The number of dimes on the nth square is 2^{n-1} (since the first square is $n = 0$ for $2^0 = 1$); therefore we will have 2^9 dimes on the 10th square.

10.5b Problem: How many dimes will you have stacked on the nth square?

10.5b Solution: There will be 2^{n-1} dimes stacked on the nth square.

10.5c Problem: How many dimes will you have stacked on the 64th square?

10.5c Solution: Using our formula, there will be 2^{63} dimes on the 64th dime.

10.5d Problem: Assuming a dime is 1 mm thick, how high will this last pile be?

10.5d Solution: There are 2^{63} dimes in the 64th page, and since each one is 1 mm thick, the last pile will be 2^{63} mm thick.

10.5e Problem: The distance from the earth to the sun is approximately 150 million km. Relate the height of the last pile of dimes to this distance.

10.5e Solution: 1 millimeter is 0.000001 kilometer; therefore the dime is $0.000001\left(2^{63}\right) = 9223372036854.775$ kilometers. The distance from the earth to the sun is $150 \times 10^6 = 1.5 \times 150000000$ kilometers, which is much smaller than the height of the dimes.

4 Question 10.6

10.6 Problem: Myoglobin and hemoglobin are oxygen carrying molecules in the human body. Hemoglobin is found inside red blood cells, which flow from the lungs to the muscles through the bloodstream. Myoglobin is found in muscle cells. The function

$$Y = M(p) = \frac{p}{1+p}$$

calculates the fraction of myoglobin saturated with oxygen at a given pressure p torrs. For example, at a pressure of 1 torr, $M(1) = 0.5$, which means half of the myoglobin (i.e. 50%) is oxygen saturated. (Note: More precisely, you need to use something called the "partial pressure", but the distinction is not important for this problem.) Likewise, the function

$$Y = H(p) = \frac{p^{2.8}}{26^{2.8} + p^{2.8}}$$

calculates the fraction of hemoglobin saturated with oxygen at a given pressure p.

10.6a Problem: The graphs of $M(p)$ and $H(p)$ are given below on the domain $0 \le p \le 100$; which is which?

10.6a Solution: The diagram given in the book displays two curves - one that sticks to the left and top of the graph like a good AUC score, and another that looks sigmoid-shaped. I suspect these curves are $M(p)$ and $H(p)$, respectively. The equation for the sigmoid function is $y = \frac{1}{1+e^{-x}}$; the exponentiation shares resemblance with $H(p)$. Furthermore, I would expect inverse function-like behavior more from a function like $M(p)$ rather than $H(p)$. Furthermore, testing in a sample value suggests $M(p)$ is the function that rises more quickly.

10.6b Problem: If the pressure in the lungs is 100 torrs, what is the level of oxygen saturation of the hemoglobin in the lungs?

10.6b Solution: Calculating $p = 100$:

$$H(100) = \frac{100^{2.8}}{26^{2.8} + 100^{2.8}} = 0.97750\ldots$$

The level of oxygen saturation of hemoglobin in the lungs at 100 torrs of pressure is $\approx 97.750\%$.

10.6c Problem: The pressure in an active muscle is 20 torrs. What is the level of oxygen saturation of myoglobin in an active muscle? What is the level of hemoglobin in an active muscle?

10.6c Solution: Calculating $p = 20$ for $M(p)$ and $H(p)$:

$$M(20) = \frac{20}{1+20} = 0.95238\ldots$$

$$H(20) = \frac{20^{2.8}}{26^{2.8} + 20^{2.8}} = 0.32418\ldots$$

The level of oxygen saturation is

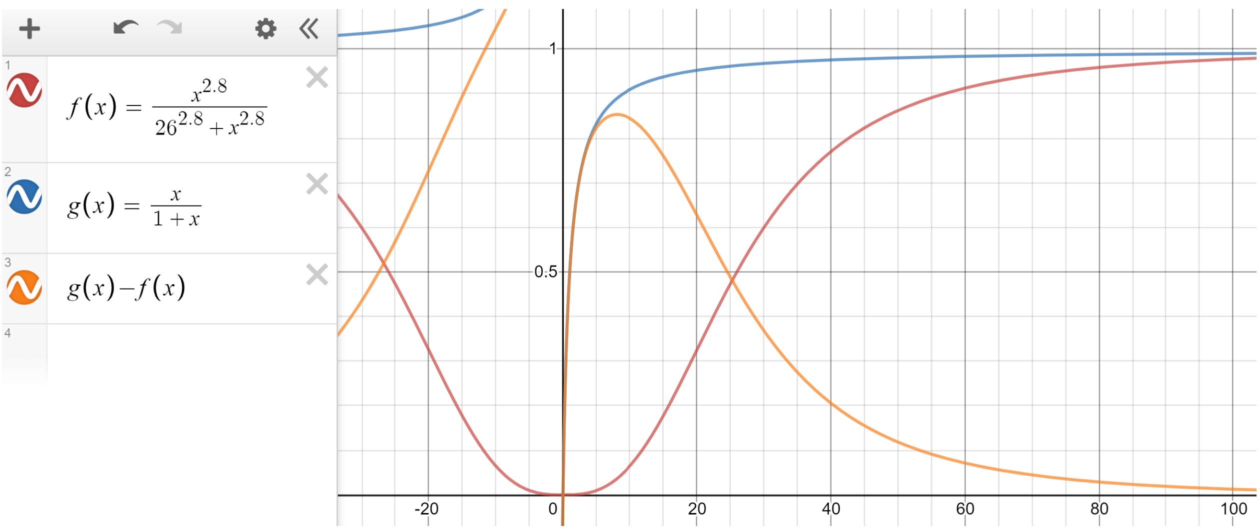
- $\approx 95.238\%$ for myoglobin;
- $\approx 0.32418\%$ for hemoglobin.

10.6d Problem: Define the efficiency of oxygen transport at a given pressure p to be $M(p) - H(p)$. What is the oxygen transport efficiency at 20 torrs? At 40 torrs? At 60 torrs? Sketch the graph of $M(p) - H(p)$; are there conditions under which transport efficiency is maximized (explain)?

10.6d Solution: Plugging in values for p, the efficiency of the oxygen transport is

- 0.62819% at 20 torrs;
- 0.20598% at 40 torrs;
- 0.07135% at 60 torrs.

Visualizing:



The transport efficiency is maximized at about 8 torrs - the peak of its graph for domain $0 \leq p \leq 100$. This is when $M(p)$ rises quickly enough but $H(p)$ rises slowly enough such that their difference is maximized.

5 Additional Problem Set Question 1

Problem: Write the function

$$f(x) = \frac{9^x \left(2^{6x-3} + 3^{x-3}\sqrt{25^{3x-4}}\right)}{3 \cdot \sqrt{49^x} \cdot 7^{-x}}$$

Solution:

$$
\begin{aligned}
f(x) &= \frac{9^x\left(2^{6x-3}+3^{x-3}\sqrt{25^{3x-4}}\right)}{3\cdot\sqrt{49^x}\cdot 7^{-x}}\\
&= \frac{9^x\left(2^{6x-3}+3^{x-3}\cdot 5^{3x-4}\right)}{3\cdot 7^x\cdot\left(\frac{1}{7}\right)^x}\\
&= \frac{9^x\left(2^{6x-3}+3^{x-3}\cdot 5^{3x-4}\right)}{3\cdot 1}\\
&= 9^x\left(\frac{16875\cdot 2^{6x}}{135000}+\frac{3^x}{27}\cdot\frac{5^{3x}}{625}\right)\left(\frac{1}{3}\right)\\
&= 9^x\left(\frac{16875\cdot 2^{6x}+3^x\cdot 5^{3x}}{135000}\right)\left(\frac{1}{3}\right)\\
&= \left(\frac{16875\cdot 9^x\cdot 2^{6x}+3^x\cdot 5^{3x}\cdot 9^x}{3\cdot 135000}\right)\\
&= \frac{16875\cdot 9^x\cdot 64^x+3^x\cdot 125^x\cdot 9^x}{3\cdot 135000}\\
&= \frac{16875\cdot(9\cdot 64)^x+(3\cdot 125\cdot 9)^x}{3\cdot 135000}
\end{aligned}
$$

This solution, although it is certainly not clean, has two exponentiated terms ('exponentials') (and for that matter each exponent is as small as it can get). I believe one can only reduce it to two exponents because of the '+' between the $16875\cdot(9\cdot 64)^x$ term and the $(3\cdot 125\cdot 9)^x$. It is not possible to merge them together into one exponent, because they have different bases and are added - not multiplied - together.

6 Additional Problem Set Question 2

Problem: Write the function $g(x) = 64^{\frac{3x+7}{2}+\frac{4-5x}{3}}$ so that the expressions are as simple as possible. Defend your answer - what leads you to the conclusion that this is the best possible?

Solution:

$$
\begin{aligned}
g(x) &= 64^{\frac{3x+7}{2}+\frac{4-5x}{3}}\\
&= 64^{\frac{9x+21}{6}+\frac{8-10x}{6}}\\
&= 64^{\frac{29-x}{6}}\\
&= \sqrt[6]{64}^{29-x}\\
&= 2^{(29-x)}\\
&= 2^{29}\cdot\left(\frac{1}{2}\right)^x\\
&= 36870912\cdot\left(\frac{1}{2}\right)^x
\end{aligned}
$$

This may not be the *best solution*, but it certainly satisfies the criteria for being a very good solution; it is in standard exponent form, and the terms involved - 36870912 and $\frac{1}{2}$ - are not especially intricate, being an integer and a simple fraction.

7 Additional Problem Set Question 3

Problem: Is $f(x) = x^x$ an exponential function? Why or why not?

Solution: No, for the simple reason that we define the base to be a number, and that it must be constant. Furthermore, if x is permitted to be a base, it can become negative, which again violates a constraint of exponential functions.

8 Additional Problem Set Question 4

Problem: Solve the equation $2^{3x-6} = 8^{4-x^2}$

Solution:

$$\begin{aligned}
2^{3x-6} &= 8^{4-x^2} \\
2^{3x-6} &= 2^{12-3x^2} \\
3x-6 &= 12-3x^2 \\
3x^2+3x-18 &= 0 \\
x^2+x-6 &= 0 \\
(x+3)(x-2) &= 0 \\
x &= -3, 2
\end{aligned}$$

Therefore, $x = -3$ and $x = 2$.