Collingwood 45

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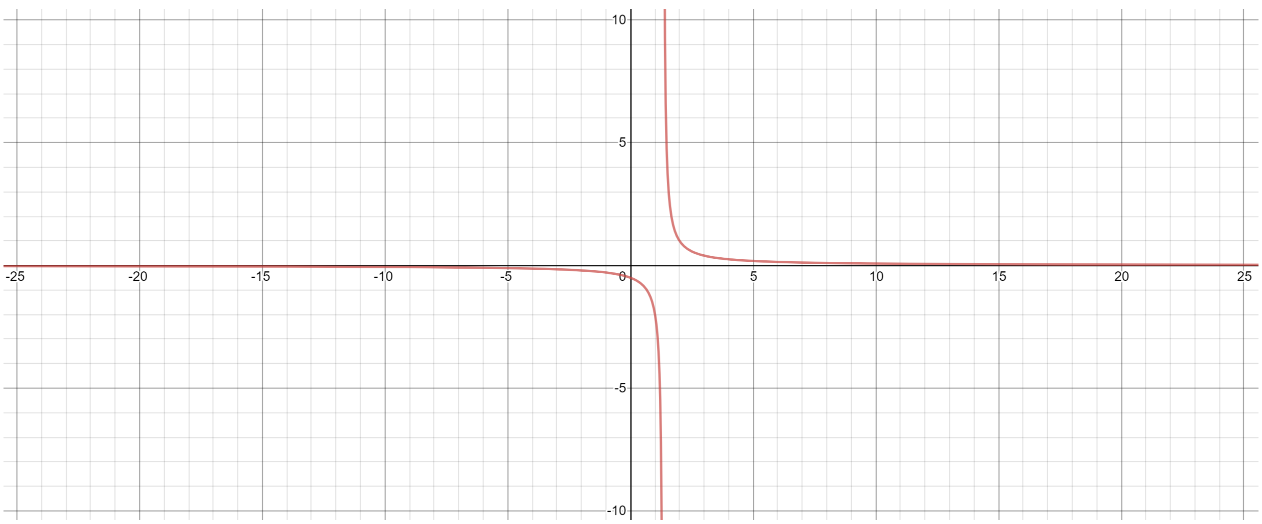
1 Question 9.1

9.1 Context: Let $f(x) = \frac{2}{3x-4}$ on the largest domain for which the formula makes sense.

9.1a Problem: Find the domain and range of $f(x)$, then sketch the graph.

9.1a Solution: All values of x can be put into $f(x)$, except values that make the denominator equal zero; solving this linear equation $3x - 4 = 0$ yields $x = \frac{4}{3}$. Therefore, the domain is $(-\infty, \frac{4}{3}) \cup (\frac{4}{3}, \infty)$. $f(x)$ can output any value, except for $f(x) = 0$, since one cannot divide 2 by anything to achieve 0. Hence, the range is $(-\infty, 0) \cup (0, \infty)$. Graphing $f(x)$ confirms this.



9.1b Problem: Find the domain, range and rule for the inverse function f^{-1}, then sketch its graph.

9.1b Solution: Finding the inverse function by switching x and y:

$$x = \frac{2}{3y-4}$$
$$3y - 4 = \frac{2}{x}$$
$$y = \frac{4}{3} + \frac{2}{3x}$$

Hence, $f^{-1}(x) = \frac{4}{3} + \frac{2}{3x}$. Since an inverse function 'swaps' the x and y axes, we can simply re-appropriate the domain and ranges of f, but switch them. Therefore, the domain of f^{-1} is $(-\infty, 0) \cup (0, \infty)$ (the range of f) and the range is $(-\infty, \frac{4}{3}) \cup (\frac{4}{3}, \infty)$ (the domain of f).

2 Question 9.2

9.2 Context: Find the inverse function of each of the following functions. Specify the domains of the inverse functions.

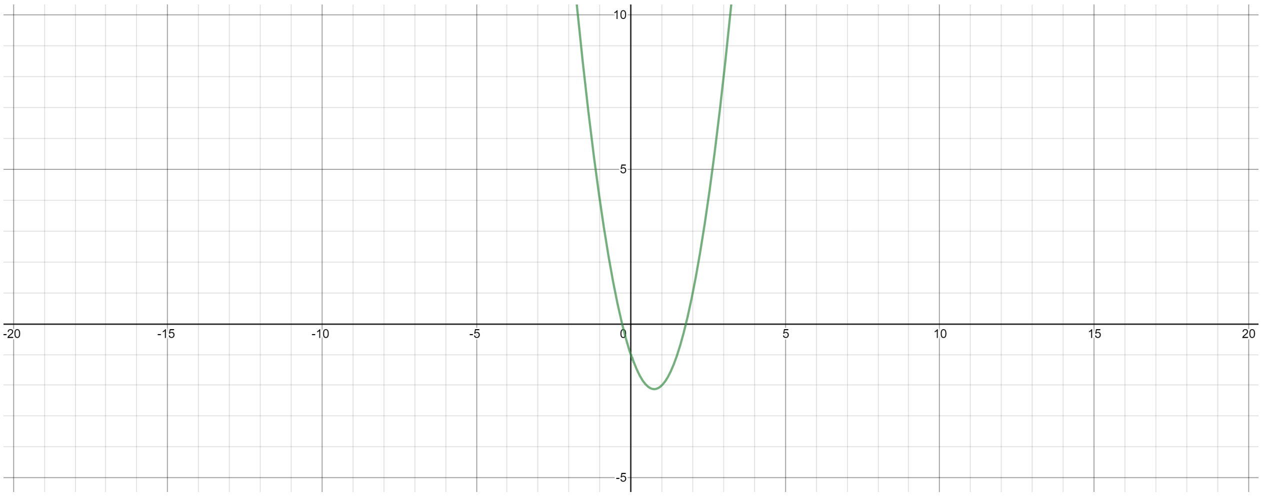
Part	Aspect	Content
a	Function	$f(x) = \frac{1}{5}x + 8$
a	Work	$x = \frac{1}{5}y + 8 \to 5x - 40 = y$
a	Inverse Function	$f^{-1}(x) = 5x - 40$
a	Domain	$(-\infty, \infty)$
b	Function	$h(x) = \frac{5}{x+3}$
b	Work	$x = \frac{5}{y+3} \to y = \frac{5}{x} - 3$
b	Inverse Function	$h^{-1}(x) = \frac{5}{x} - 3$
b	Domain	$(-\infty, 0) \cup (0, \infty)$
c	Function	$g(x) = 4\sqrt{3-x} - 7$
c	Work	$x = 4\sqrt{3-y} - 7 \to \left(\frac{7+x}{4}\right)^2 = 3 - y \to y = 3 - \left(\frac{7+x}{4}\right)^2$
c	Inverse Function	$g^{-1}(x) = 3 - \left(\frac{7+x}{4}\right)^2$
c	Domain	Range of $g(x) = [-7, \infty) \to [-7, \infty)$
d	Function	$j(x) = \sqrt{x} + \sqrt{x-1}$
d	Work	$x = \sqrt{y} + \sqrt{y-1} \to x - \sqrt{y} = \sqrt{y-1} \to x^2 - 2x\sqrt{y} = -1$ $\to -x^2 + 2x\sqrt{y} = 1 \to 2x\sqrt{y} = 1 + x^2 \to \sqrt{y} = \frac{1+x^2}{2x}$ $\to y = \left(\frac{1+x^2}{2x}\right)^2 = \frac{x^4+2x^2+1}{4x^2}$
d	Inverse Function	$j^{-1}(x) = \frac{x^4+2x^2+1}{4x^2}$
d	Domain	Range of $j(x) = [1, \infty) \to [1, \infty)$
e	Function	$k(x) = \sqrt{16 - x^2}, 0 \le x \le 4$
e	Work	$x = \sqrt{16 - y^2} \to x^2 = 16 - y^2 \to y = \sqrt{16 - x^2}$
e	Inverse Function	$k^{-1}(x) = \sqrt{16 - x^2}$
e	Domain	Range of $k(x) = [k(4), k(0)] = [0, 4] \to [0, 4]$

3 Question 9.3

9.3 Context: For this problem, $y = f(x) = 2x^2 - 3x - 1$ on the domain of all real numbers.

9.3a Problem: Sketch the function graph and find the coordinates of the vertex $P = (a, b)$.

9.3a Solution: The x-coordinate of the vertex is $-\frac{b}{2a} = \frac{3}{4}$. $f\left(\frac{3}{4}\right) = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) - 1 = -\frac{17}{8}$. Therefore, the vertex P is $\left(\frac{3}{4}, -\frac{17}{8}\right)$.



9.3b Problem: Explain why $y = f(x)$ does not have a inverse function on the domain of all real numbers.

9.3b Solution: Inverting a function is the same as swapping its x and y axes. In order for an expression to be a function, there must only be one y for every x. In $f(x)$, for all points above the vertex, there are two x-points for every y-point. When the function is inverted, there will be two y-points for every x-point, violating the definition of a function.

9.3c Problem: Restrict $y = f(x)$ to the domain $a \leq x$ and find the formula for the inverse function $f^{-1}(y)$. What are the domain and range of the inverse function?

9.3c Solution: Restricting to domain $\frac{3}{4} \leq x$, the range must be $y \leq -\frac{17}{8}$. Finding the inverse:

$$2y^2 - 3y - 1 = x$$
$$2y^2 - 3y - 1 - x = 0$$
$$y = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1-x)}}{2(2)}$$
$$y = \frac{3 \pm \sqrt{8x+17}}{4}$$

Therefore, $f^{-1}(x) = \frac{3+\sqrt{8x+17}}{4}$. However, we limited it such that the domain is $[3/4, \infty)$, and as the function is inverted, the range is $[3/4, \infty)$. We want the solution with the higher value of x since the range is bounded below and unbounded above; hence, the inverse function is $f^{-1}(y) = \frac{3+\sqrt{8y+17}}{4}$. The domain of $f^{-1}(x)$ is the range of $f(x)$, which is $[-17/8, \infty)$. The range of $f^{-1}(x)$ is the domain of $f(x)$, which is $[3/4, \infty)$.

9.3d Problem: Restrict $y = f(x)$ to the domain $x \leq a$ and find the formula for the inverse function $f^{-1}(y)$. What are the domain and range of the inverse function?

9.3d Solution: Since the domain is limited to $(-\infty, 3/4]$, the range of the inverted function is $(-\infty, 3/4]$. Therefore, we want the lower solution, which is $f^{-1}(y) = \frac{3-\sqrt{8y+17}}{4}$. The domain of $f^{-1}(x)$ is the range of $f(x)$, which is $[-17/8, \infty)$. The range of $f^{-1}(x)$ is the domain of $f(x)$, which is $(-\infty, 3/4]$.

4 Question 9.4

9.4 Context: Which of the following graphs are one-to-one? If they are not one-to-one, section the graph up into parts that are one-to-one.

9.4a-9.4b Solutions:

Part	One-to-One?	Sectioning (if not one-to-one)
a	No	Choose only the left or the right side of the function ($x \leq 0$ or $x \geq 0$).
b	Yes	
c	No	There are two extrema; section the function into one of the three buckets formed.
d	No	There is no way to section the function such that it is one-to-one.

5 Question 9.5

9.5 Problem: Show that, for every value of a, the function $f(x) = a + \frac{1}{x-a}$ is its own inverse.

9.5 Solution: Solving for the inverse of $f(x)$:

$$\begin{aligned} x &= a + \frac{1}{y-a} \\ x - a &= \frac{1}{y-a} \\ y - a &= \frac{1}{x-a} \\ y &= a + \frac{1}{x-a} \end{aligned}$$

Therefore,

$$f(x) = f^{-1}(x) = a + \frac{1}{x-a}$$

6 Question 9.6

9.6 Context: Clovis is standing at the edge of a cliff, which slopes 4 feet downward from him for every 1 horizontal foot. He launches a small model rocket from where he is standing. With the origin of the coordinate system located where he is standing, and the x-axis extending horizontally, the path of the rocket is described by the formula $y = -2x^2 + 120x$.

9.6a Problem: Give a function $h = f(x)$ relating the height h of the rocket above the sloping ground to its x-coordinate.

9.6a Solution: The ground is represented by the equation $y = -4x$. The difference between the height of the rocket and the ground is thus $-2x^2 + 120x + 4x =$$f(x) = -2x^2 + 124x$.

9.6b Problem: Find the maximum height of the rocket above the sloping ground. What is its x-coordinate when it is at its maximum height?

9.6b Solution: The vertex of $f(x)$ is $-\frac{b}{2a} = -\frac{124}{-4} = 31$. This maximum height is $f(31) =$1922 feet. This occurs at x-coordinate $x = 31$.

9.6c Problem: Clovis measures its height h of the rocket above the sloping ground while it is going up. Give a function $x = g(h)$ relating the x-coordinate of the rocket to h.

9.6c Solution: This would be the inverse of $f(x)$. Solving:

$$\begin{aligned}
-2y^2 + 124y &= x \\
-2y^2 + 124y - x &= 0 \\
y &= \frac{-124 \pm \sqrt{(124)^2 - 4(-2)(-x)}}{2(-2)} \\
&= \frac{-124 \pm 2\sqrt{3844 - 2x}}{4} \\
&= -31 \pm \frac{\sqrt{3844 - 2x}}{2}
\end{aligned}$$

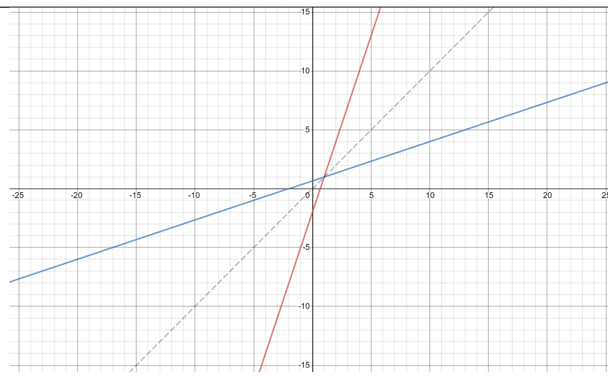
Since it's going up, the domain of $f(x)$ is $[0, 31]$, or the *smaller* values of x. This means that the range of $f^{-1}(x)$ is also the smaller value, so we choose the $-$ aspect of $\pm$; therefore $g(h) = 31 - \frac{\sqrt{3844-2x}}{2}$.

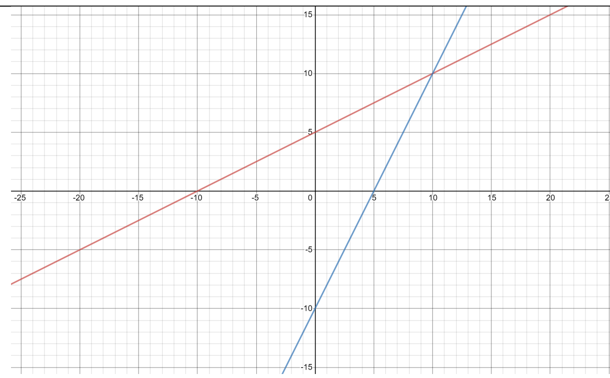
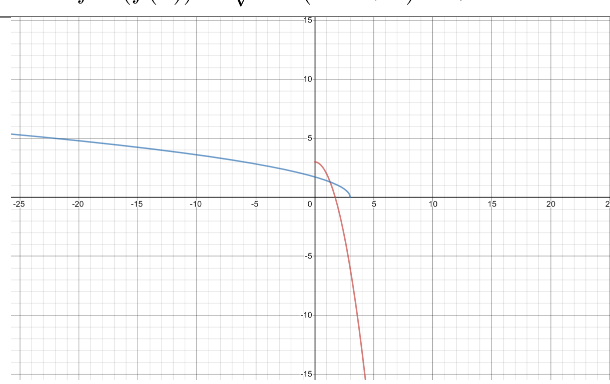
9.6d Problem: Does this function still work when the rocket is going down? Explain.

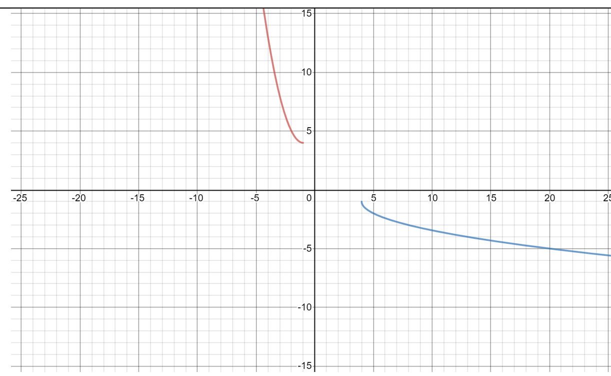
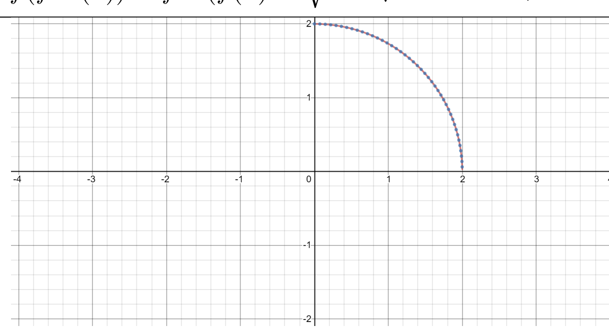
9.6d Solution: No, it doesn't still work. The shape described is a parabola, and when finding its inverse, the $\pm$ present prevents the inverse from being a function. Therefore, there must be one expression for going up and one expression for going down; if not, the function would not be one-to-one.

7 Question 9.7

9.7 Context: For each of the following functions: (1) sketch the function, (2) find the inverse function, and (3) sketch the inverse function. In each case, indicate the correct domains and ranges. (4) Finally, make sure you test each of the functions you propose as an inverse. (Images: $f(x)$ in red, $f^{-1}(x)$ in blue.)

Part	Aspect	Content
a	Function	$f(x) = 3x - 2$
a	Work	$x = 3y - 2 \rightarrow \frac{2+x}{3} = y$
a	Inverse Function	$f^{-1}(x) = \frac{2+x}{3}$
a	Domain and Range	Domain of $f(x)$: $\mathbb{R}$; Range of $f(x) : \mathbb{R}$ Domain of $f^{-1}(x) : \mathbb{R}$; Range of $f^{-1}(x) : \mathbb{R}$
a	Verification	$f(f^{-1}(x)) = 3\left(\frac{2+x}{3}\right) - 2 = 2 + x - 2 = x$ $f^{-1}(f(x)) = \frac{2+(3x-2)}{3} = \frac{3x}{3} = x$
a	Image	

b	Function	$f(x) = \frac{x}{2} + 5$
b	Work	$x = \frac{1}{2}y + 5 \to 2x - 10 = y$
b	Inverse Function	$f^{-1}(x) = 2x - 10$
b	Domain and Range	Domain of $f(x)$: $\mathbb{R}$; Range of $f(x) : \mathbb{R}$ Domain of $f^{-1}(x) : \mathbb{R}$; Range of $f^{-1}(x) : \mathbb{R}$
b	Verification	$f(f^{-1}(x)) = \frac{2x-10}{2} + 5 = x - 5 + 5 = x$ $f^{-1}(f(x)) = 2\left(\frac{x}{2} + 5\right) - 10 = x + 10 - 10 = x$
b	Image	
c	Function	$f(x) = -x^2 + 3, x \geq 0$
c	Work	$x = -y^2 + 3 \to y = \sqrt{3 - x}$ (+ instead of $-$ because $x \geq 0$)
c	Inverse Function	$f^{-1}(x) = y = \sqrt{3 - x}$
c	Domain and Range	Domain of $f(x)$: $\mathbb{R}$; Range of $f(x) : (-\infty, 3]$ Domain of $f^{-1}(x) : (-\infty, 3]$; Range of $f^{-1}(x) : \mathbb{R}$
c	Verification	$f(f^{-1}(x)) = -(\sqrt{3 - x})^2 + 3 = -3 + x + 3 = x$ $f^{-1}(f(x)) = \sqrt{3 - (-x^2 + 3)} = \sqrt{x^2} = x$
c	Image	

d	Function	$f(x) = x^2 + 2x + 5, x \leq -1$
d	Work	$y^2 + 2y + 5 - x = 0 \to y = \frac{-2-\sqrt{2^2-4(5-x)}}{2} = -\sqrt{x-4} - 1$ $-$ is used instead of $+$ b/c $x \leq -1 \to y \leq -1$ for the inverse function, meaning that the lower value is favored.
d	Inverse Function	$f^{-1}(x) = -\sqrt{x-4} - 1$
d	Domain and Range	Domain of $f(x)$: $x \leq -1$; Range of $f(x) : [f(-1), \infty) \to [4, \infty)$ Domain of $f^{-1}(x) : (4, \infty)$; Range of $f^{-1}(x) : x \leq -1$
d	Verification	$f(f^{-1}(x)) = \left(-\sqrt{x-4} - 1\right)^2 + 2\left(-\sqrt{x-4} - 1\right) + 5 = x$ $f^{-1}(f(x)) = -\sqrt{x^2 + 2x + 5 - 4} - 1 = -\sqrt{(x+1)^2} - 1 = -(-(x+1)) - 1 = x$
d	Image	
e	Function	$f(x) = \sqrt{4 - x^2}, 0 \leq x \leq 2$
e	Work	$x = \sqrt{4 - y^2} \to x^2 = 4 - y^2 \to y = \sqrt{4 - x^2}$
e	Inverse Function	$f^{-1}(x) = \sqrt{4 - x^2}$
e	Domain and Range	Domain of $f(x)$: $[0, 2]$; Range of $f(x) : [0, 2]$ Domain of $f^{-1}(x) : [0, 2]$; Range of $f^{-1}(x) : [0, 2]$
e	Verification	$f(f^{-1}(x)) = f^{-1}(f(x) = \sqrt{4 - \sqrt{4 - x^2}^2} = \sqrt{x^2} = x$
e	Image	

8 Question 9.8

9.8 Context: A trough has a semicircular cross section with a radius of 5 feet. Water starts flowing into the trough in such a way that the depth of the water is increasing at a rate of 2 inches per hour.

9.8a Problem: Give a function $w = f(t)$ relating the width w of the surface of the water to the time t, in hours. Make sure to specify the domain and compute the range too.

9.8a Solution: The height of the water is $\frac{1}{6}t$ feet, where t is the number of hours that have passed. Let the trough be modelled by $x^2 + (y-5)^2 = 25, 0 \leq y \leq 5$ (since it is a semi-circle with radius 5 feet). Solving for the x-values when y, the height, equals $\frac{1}{6}t$:

$$\begin{aligned} x^2 + \left(\frac{1}{6}t - 5\right)^2 &= 25 \\ x^2 &= 25 - \left(\frac{1}{6}t - 5\right)^2 \\ x &= \pm\sqrt{25 - \left(\frac{1}{6}t - 5\right)^2} \end{aligned}$$

Therefore, the width must be two times this, or $f(t) = 2\sqrt{25 - \left(\frac{1}{6}t - 5\right)^2}$. The range is $[0, 10]$, since the minimum and maximum widths are 0 and 10, respectively. Solving for the value of t when $f(t) = 10$:

$$\begin{aligned} 10 &= 2\sqrt{25 - \left(\frac{1}{6}t - 5\right)^2} \\ 5 &= \sqrt{25 - \left(\frac{1}{6}t - 5\right)^2} \\ 25 &= 25 - \left(\frac{1}{6}t - 5\right)^2 \\ \left(\frac{1}{6}t - 5\right)^2 &= 0 \\ \frac{1}{6}t - 5 &= 0 \\ \frac{t}{6} &= 5 \\ t &= 30 \end{aligned}$$

Since we know that $t = 0$ when $f(t) = 0$, the domain is $[0, 30]$.

9.8b Problem: After how many hours will the surface of the water have width of 6 feet?

9.8b Solution: Solving for $f(t) = 6$:

$$\begin{aligned}
6 &= 2\sqrt{25 - \left(\frac{1}{6}t - 5\right)^2} \\
3 &= \sqrt{25 - \left(\frac{1}{6}t - 5\right)^2} \\
9 &= 25 - \left(\frac{1}{6}t - 5\right)^2 \\
\left(\frac{1}{6}t - 5\right)^2 &= 16 \\
\frac{1}{6}t - 5 &= \pm 4 \\
\frac{1}{6}t &= 1 \text{ or } 9 \\
t &= 6 \text{ or } 54
\end{aligned}$$

Since 54 hours is outside the domain, it takes 6 hours for the width to be 6 feet.

9.8c Problem: Give a function $t = f^{-1}(w)$ relating the time to the width of the surface of the water. Make sure to specify the domain and compute the range too.

9.8c Solution: Solving for the inverse, where x represents the width w and y represents the time t.

$$\begin{aligned}
x &= 2\sqrt{25 - \left(\frac{1}{6}y - 5\right)^2} && (1) \\
\frac{x}{2} &= \sqrt{25 - \left(\frac{1}{6}y - 5\right)^2} && (2) \\
\left(\frac{x}{2}\right)^2 &= 25 - \left(\frac{1}{6}y - 5\right)^2 && (3) \\
\left(\frac{1}{6}y - 5\right)^2 &= 25 - \left(\frac{x}{2}\right)^2 && (4) \\
\frac{1}{6}y - 5 &= -\sqrt{25 - \left(\frac{x}{2}\right)^2} && (5) \\
\frac{1}{6}y &= -\sqrt{25 - \left(\frac{x}{2}\right)^2} + 5 && (6) \\
y &= -6\sqrt{25 - \left(\frac{x}{2}\right)^2} + 30 && (7)
\end{aligned}$$

Note: on line 5, $-$ instead of $+$ is used because the domain of f is $[0, 30]$, so the range of f^{-1} is $[0, 30]$. That is, it is on the *lower side*.

Therefore, $t = f^{-1}(w) = -6\sqrt{25 - \left(\frac{w}{2}\right)^2} + 30$.

9 Question 9.9

9.9 Context: A biochemical experiment involves combining together two protein extracts. Suppose a function $\phi(t)$ monitors the amount (nanograms) of extract A remaining at time t (nanoseconds). Assume you know these facts: the function ϕ is invertible; i.e., it has an inverse function.

t	$\phi(t)$
0	6
1	5
2	3
3	1
4	0.5
10	0

9.9a Problem: At what time do you know there will be 3 nanograms of extract A remaining?

9.9a Solution: At time $t = 2$, since we know that $\phi(2) = 3$ and the function is invertible, so $t = 2$ is the *only* value of t for which $\phi(t) = 3$.

9.9b Problem: What is $\phi^{-1}(0.5)$ and what does it tell you?

9.9b Solution: $\phi^{-1}(0.5) = 4$, from looking at the table. This means that the amount of extract A remaining reaches 0.5 nanograms at after 4 nanoseconds.

9.9c Problem: (True or False) There is exactly one time when the amount of extract A remaining is 4 nanograms.

9.9c Solution: True; this is a property of an invertible function.

9.9d Problem: Calculate $\phi(\phi^{-1}(1)) =$

9.9d Solution: $\phi(\phi^{-1}(1)) = 1$, because $f(f^{-1}(x)) = x$.

9.9e Problem: Calculate $\phi^{-1}(\phi(6)) =$

9.9e Solution: $\phi^{-1}(\phi(6)) = 6$, because $f^{-1}(f(x)) = x$.

9.9f Problem: What is the domain and range of ϕ?

9.9f Solution:

- The domain is $[0, 10]$, since the time cannot be before 0 seconds, and at 10 seconds the amount of extract A remaining is 0, and any further time would likely predict negative amounts of extract remaining. Of course, this is an assumption; the domain could also be $[0, \infty)$, where after $t = 10$, $\phi(t) = 0$.
- The range is $[0, 6]$. Assuming that ϕ never increases past its initial value, the maximum value is its initial value, at 6 nanograms. On the other hand, the minimum value is its final value, at 0 nanograms.