Collingwood 44

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3 January 2021

1 Question 8.1

8.1 Context. For this problem, $f(t) = t - 1$, $g(t) = -t - 1$ and $h(t) = |t|$.

8.1a Problem: Compute the multipart rules for $h(f(t))$ and $h(g(t))$ and sketch their graphs.

8.1a Solution: $h(f(t)) = |t-1|$. When $t > 1$, the expression $t-1$ is positive. When $t < 1$, the expression $t - 1$ is negative. To write this as a multipart function, we negate the expression $t - 1$ when $t < 1$, yielding $-t + 1$. Therefore, the multipart function is

$$h(f(t)) = \begin{cases} t - 1 & \text{if } t > 1 \\ -t + 1 & \text{if } t \leq 1 \end{cases}$$

$h(g(t)) = | - t - 1|$. When $t < -1$, $-t - 1$ is positive. When $t > -1$, $-t - 1$ is negative. To write this as a multipart function, we negate $-t - 1$ when $t > -1$, yielding $t + 1$. Therefore, the multipart function is

$$h(g(t)) = \begin{cases} -t - 1 & \text{if } t < -1 \\ t + 1 & \text{if } t \geq -1 \end{cases}$$

8.1b Problem: Compute the multipart rules for $f(h(t))$ and $g(h(t))$ and sketch their graphs.

8.1b Solution: $f(h(t)) = |t| - 1$. The absolute value function $|t|$ can be written as "return t if $t > 0$, return $-t$ if $t \leq 0$. Therefore, the multipart function for $|t| - 1$ is similarly

$$f(h(t)) = \begin{cases} t - 1 & \text{if } t > 0 \\ -t - 1 & \text{if } t \leq 0 \end{cases}$$

$g(h(t)) = -|t| - 1$. Using the multipart definition of an absolute value function as found previously and substituting, the multipart function for this is

$$f(h(t)) = \begin{cases} -t - 1 & \text{if } t > 0 \\ t - 1 & \text{if } t \leq 0 \end{cases}$$

8.1c Problem: Compute the multipart rule for $h(h(t) - 1)$ and sketch the graph.

8.1c Solution: $h(h(t) - 1) = ||t| - 1|$. Using the multipart definition of an absolute value function as found previously and substituting:

$$h(h(t) - 1) = \begin{cases} |t - 1| & \text{if } t > 0 \\ | - t - 1| & \text{if } t \leq 0 \end{cases}$$

$|t-1|$ is positive when $t>1$ and negative when $t<1$; therefore it can be written as following (negating the expression when it becomes negative to make it positive), taking into account the $t>0$ restriction:

$$|t-1| = \begin{cases} t-1 & \text{if } t>1 \\ -t+1 & \text{if } 0<t\leq 1 \end{cases}$$

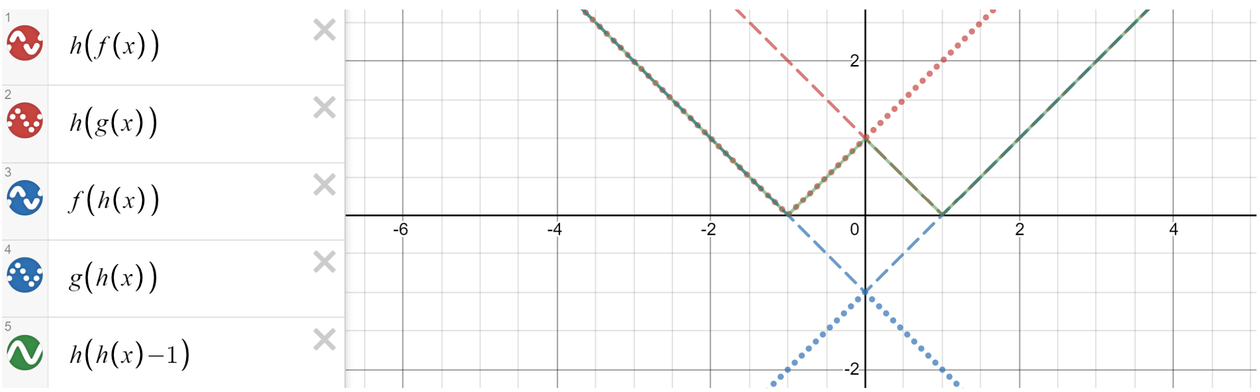
Similarly, $|-t-1|$ is positive when $t<-1$ and negative when $t>-1$; therefore it can be written as following (negating the expression when it becomes negative to make it positive), taking into account the $t\leq 0$ restriction:

$$|-t-1| = \begin{cases} -t-1 & \text{if } t<-1 \\ t+1 & \text{if } 0\geq t\geq -1 \end{cases}$$

Putting all this together:

$$h(h(t)-1) = \begin{cases} t-1 & \text{if } t>1 \\ -t+1 & \text{if } 0<t\leq 1 \\ -t-1 & \text{if } t<-1 \\ t+1 & \text{if } 0\geq t\geq -1 \end{cases}$$

Graph sketches are here.



2 Question 8.2

8.2 Context: Write each of the following functions as a composition of two simpler functions: (There is more than one correct answer.)

8.2a-8.2f Problems and Solutions: Composite functions are defined as $f(x)$ and $g(x)$ such that $f(g(x))$ is equivalent to the function provided in the problem.

Part	Function	Composite Function
a	$y = (x-11)^5$	$g(x) = x - 11, f(x) = x^5$
b	$y = \sqrt[3]{1+x^2}$	$g(x) = x^2, f(x) = \sqrt[3]{1+x}$
c	$y = 2(x-3)^5 - 5(x-3)^2 + \frac{1}{2}(x-3) + 11$	$g(x) = x - 3, f(x) = 2x^5 - 5x^2 + \frac{1}{2}x + 11$
d	$y = \frac{1}{x^2+3}$	$g(x) = x^2 + 3, f(x) = \frac{1}{x}$
e	$y = \sqrt{\sqrt{x}+1}$	$g(x) = \sqrt{x} + 1, f(x) = \sqrt{x}$
f	$y = 2 - \sqrt{5-(3x-1)^2}$	$g(x) = \sqrt{5-(3x-1)^2}, f(x) = 2 - x$

3 Question 8.3

8.3a Problem: Let $f(x)$ be a linear function, $f(x) = ax + b$ for constants a and b. Show that $f(f(x))$ is a linear function.

8.3a Solution:

$$f(f(x)) = a(ax+b) + b = a^2x + ab + b$$

Since $a^2x + ab + b$ is linear, $f(f(x))$ is a linear function.

8.3b Problem: Find a function $g(x)$ such that $g(g(x)) = 6x - 8$.

8.3b Solution: We derived the expansion of $f(f(x))$ in part a where $f(x) = ax+b$. Let us set this equal to $6x - 8$ and attempt to find a and b where $g(x) = ax + b$.

$$a^2x + ab + b = 6x - 8$$

We know that a must be $\sqrt{6}$, since it is the coefficient of x. Plugging this in:

$$6x + \sqrt{6}b + b = 6x - 8$$
$$\sqrt{6}b + b = -8$$
$$(\sqrt{6}+1)b = -8$$
$$b = \frac{-8}{\sqrt{6}+1}$$

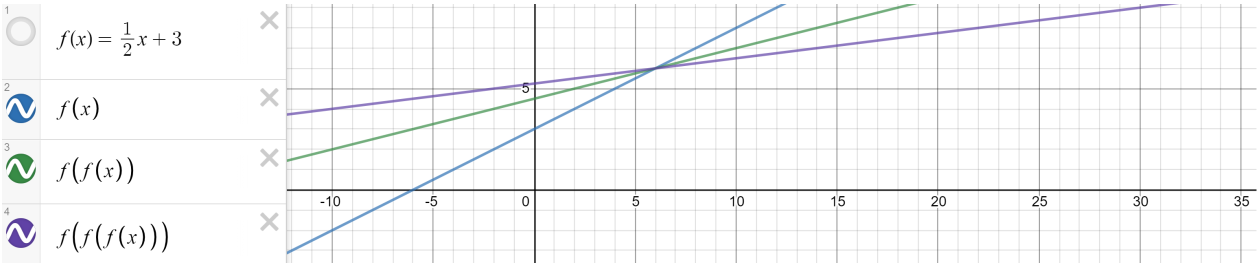
Therefore, $g(x) = \sqrt{6}x - \frac{8}{\sqrt{6}+1}$.

4 Question 8.4

8.4 Context: Let $f(x) = \frac{1}{2}x + 3$.

8.4a Problem: Sketch the graphs of $f(x), f(f(x)), f(f(f(x)))$ on the interval $-2 \leq x \leq 10$.

8.4a Solution:



8.4b Problem: Your graphs should all intersect at the point (6,6). The value $x = 6$ is called a fixed point of the function $f(x)$ since $f(6) = 6$; that is, 6 is fixed - it doesn't move when f is applied to it. Give an explanation for why 6 is a fixed point for any function $f(f(f(...f(x)...)))$.

8.4b Solution: The inputs and outputs of the function are identical; since $f(6) = \frac{1}{2} \cdot 6 + 3 = 3 + 3 = 6$, for this specific value of x, $f(x)$ acts like multiplying the input by 1. Multiplying 6 by 1 an arbitrary number of times still yields 6.

8.4c Problem: Linear functions (with the exception of $f(x) = x$) can have at most one fixed point. Quadratic functions can have at most two. Find the fixed points of the function $g(x) = x^2 - 2$.

8.4c Solution: We are looking for a value of x such that the inputs and outputs are the same; therefore, $g(x) = x$. Solving for this:

$$x^2 - 2 = x \to x^2 - x - 2 = 0 \to (x-2)(x+1) = 0 \to x = 2, -1$$

Therefore, the fixed points for $g(x) = x^2 - 2$ are $x = 2$ and $x = -1$.

8.4d Problem: Give a quadratic function whose fixed points are $x = -2$ and $x = 3$.

8.4d Solution: We need a quadratic function $f(x) = ax^2 + bx + c$ such that $f(-2) = -2$ and $f(3) = 3$. That is,

$$f(-2) = 4a - 2b + c = -2$$

$$f(3) = 9a + 3b + c = 3$$

Subtracting the two yields

$$5a + 5b = 5 \to a + b = 1$$

To keep things simple, let us settle with $a = 1$ and $b = 0$. We need to solve for c.

$$4a - 2b + c = -2 \to 4 + c = -2 \to c = -6$$

Therefore, a quadratic with fixed points at $x = -2$ and $x = 3$ is $f(x) = x^2 - 6$.

5 Question 8.5

8.5 Context: A car leaves Seattle heading east. The speed of the car in mph after m minutes is given by the function $C(m) = \frac{70m^2}{10+m^2}$.

8.5a Problem: Find a function $m = f(s)$ that converts seconds s into minutes m. Write out the formula for the new function $C(f(s))$; what does this function calculate?

8.5a Solution: To find the number of minutes in a quantity of seconds, we divide that quantity by 60. Therefore, $m = f(s) = \frac{s}{60}$. Hence,

$$C(f(s)) = \frac{70(\frac{s}{60})^2}{10 + (\frac{s}{60})^2}$$

This function calculates the speed of a car, in mph, after s seconds.

8.5b Problem: Find a function $m = g(h)$ that converts hours h into minutes m. Write out the formula for the new function $C(g(h))$; what does this function calculate?

8.5b Solution: To find the number of minutes in a quantity of hours, we multiply that quantity by 60. Therefore, $m = g(h) = 60h$. Hence,

$$C(f(s)) = \frac{70(60h)^2}{10 + (60h)^2}$$

This function calculates the speed of a car, in mph, after h hours.

8.5c Problem: Find a function $z = v(s)$ that converts mph s into ft/sec z. Write out the formula for the new function $v(C(m)$; what does this function calculate?

8.5c Solution: Let us write s in terms of ft/sec.

$$s\frac{\text{miles}}{\text{hour}} = 5280s\frac{\text{feet}}{\text{hour}} = 88s\frac{\text{feet}}{\text{minute}} = \frac{22}{15}s\frac{\text{feet}}{\text{sec}}$$

Therefore, $v(s) = \frac{22}{15}s$. Hence,

$$z(C(m)) = \frac{22}{15}\left(\frac{70m^2}{10 + m^2}\right)$$

This function calculates the speed of a car, in ft/sec., after m minutes.

6 Question 8.6

8.6 Context: Compute the compositions $f(g(x))$, $f(f(x))$ and $g(f(x))$ in each case:

Part	Aspect	Solution/Content
a	Problem	$f(x) = x^2, g(x) = x + 3$
a	$f(g(x))$	$f(x + 3) = (x + 3)^2$
a	$f(f(x))$	$f(x^2) = x^4$
a	$g(f(x))$	$g(x^2) = x^2 + 3$
b	Problem	$f(x) = \frac{1}{x}, g(x) = \sqrt{x}$
b	$f(g(x))$	$f(\sqrt{x}) = \frac{1}{\sqrt{x}}$
b	$f(f(x))$	$f(\frac{1}{x}) = x$
b	$g(f(x))$	$g\left(\frac{1}{x}\right) = \sqrt{\frac{1}{x}}$

Part	Aspect	Solution/Content
c	Problem	$f(x) = 9x + 2, g(x) = \frac{1}{9}(x - 2)$
c	$f(g(x))$	$f\left(\frac{1}{9}(x-2)\right) = x$
c	$f(f(x))$	$f(9x + 2) = 81x + 18 + 2 = 81x + 20$
c	$g(f(x))$	$g(9x + 2) = x$
d	Problem	$f(x) = 6x^2 + 5, g(x) = x - 4$
d	$f(g(x))$	$f(x - 4) = 6(x - 4)^2 + 5$
d	$f(f(x))$	$f(6x^2 + 5) = 6(6x^2 + 5)^2 + 5$
d	$g(f(x))$	$g(6x^2 + 5) = 6x^2 + 1$
e	Problem	$f(x) = 4x^3 - 3, g(x) = \sqrt[3]{2x + 6}$
e	$f(g(x))$	$f(\sqrt[3]{2x + 6}) = 8x + 24 - 3 = 8x + 21$
e	$f(f(x))$	$f(4x^3 - 3) = 4(4x^3 - 3)^3 - 3$
e	$g(f(x))$	$g(4x^3 - 3) = \sqrt[3]{2(4x^3 - 3) + 6} = \sqrt[3]{8x^3} = 2x$
f	Problem	$f(x) = 2x + 1, g(x) = x^3$
f	$f(g(x))$	$f(x^3) = 2x^3 + 1$
f	$f(f(x))$	$f(2x + 1) = 2(2x + 1) + 1 = 4x + 3$
f	$g(f(x))$	$g(2x + 1) = (2x + 1)^3$
g	Problem	$f(x) = 3, g(x) = 4x^2 + 2x + 1$
g	$f(g(x))$	$f(4x^2 + 2x + 1) = 3$
g	$f(f(x))$	$f(3) = 3$
g	$g(f(x))$	$g(3) = 4(9) + 2(3) + 1 = 43$
h	Problem	$f(x) = -4, g(x) = 0$
h	$f(g(x))$	$f(0) = -4$
h	$f(f(x))$	$f(-4) = -4$
h	$g(f(x))$	$g(-4) = 0$

7 Question 8.7

8.7 Problem: Let $y = f(z) = \sqrt{4 - z^2}$ and $z = g(x) = 2x + 3$. Compute the composition $y = f(g(x))$. Find the largest possible domain of x-values so that the composition $y = f(g(x))$ is defined.

8.7 Solution: Computing the composition -

$$f(g(x)) = f(2x + 3) = \sqrt{4 - (2x + 3)^2} = \sqrt{-4x^2 - 12x - 5}$$

The square root function is undefined for real numbers when a in $\sqrt{a}$ is negative. Therefore, we need to find values of x such that

$$-4x^2 - 12x - 5 > 0 \rightarrow 4x^2 + 12x + 5 > 0$$

Finding the roots:

$$\begin{aligned} 4x^2 + 12x + 5 &= 0 \\ 4x^2 + 10x + 2x + 5 &= 0 \\ 2x(2x + 5) + 1(2x + 5) &= 0 \\ (2x + 1)(2x + 5) &= 0 \\ 2x + 1 &= 0 \\ x &= -\frac{1}{2} \\ 2x + 5 &= 0 \\ x &= -\frac{5}{2} \end{aligned}$$

Testing an arbitrary value $x = 0$, $4x^2 + 12x + 5 = 5$. Hence, $-4x^2 - 12x - 5 > 0$ on the outsides of the two roots, meaning that $-4x^2 - 12x - 5 < 0$ on the inside of the two roots.

Therefore, all other regions, being $-\frac{5}{2} \leq x \leq -\frac{1}{2}$, are defined for the composite function $\sqrt{-4x^2 - 12x - 5}$.

8 Question 8.8

8.8 Context: Suppose you have a function $y = f(x)$ such that the domain of $f(x)$ is $1 \leq x \leq 6$ and the range of $f(x)$ is $-3 \leq y \leq 5$.

8.8a Problem: What is the domain of $f(2(x - 3))$?

8.8a Solution: One can find the domain of $f(2(x - 3))$ by finding which inputs of x yield an output for the domain of $f(x)$ when transformed through $2(x - 3)$. That is,

$$2(x - 3) = 1$$

$$2(x - 3) = 6$$

Solving these yields $x = \frac{7}{2}$ and $x = 6$, respectively. Therefore, the domain of $f(2(x - 3))$ is $\frac{7}{2} \leq x \leq 6$.

8.8b Problem: What is the range of $f(2(x - 3))$?

8.8b Solution: The range of $2(x - 3)$ is $\mathbb{R}$. The range of $f(x)$ is $-3 \leq y \leq 5$. Therefore, the range of $f(2(x - 3))$ is $-3 \leq y \leq 5$.

8.8c Problem: What is the domain of $2f(x) - 3$?

8.8c Solution: The domain of $f(x)$ is $1 \leq x \leq 6$. Multiplying $f(x)$ by 2 or subtracting 3 from it only impact the *output*, or y, not the domain. Therefore, the domain of $2(f(x)) - 3$ remains constant at $1 \leq x \leq 6$.

8.8d Problem: What is the range of $2f(x) - 3$?

8.8d Solution: The range of $f(x)$ is $-3 \leq f(x) \leq 5$. When the function is multiplied by 2, the range is multiplied by 2 as well; therefore it becomes $-6 \leq 2f(x) \leq 10$. Applying the same logic, $-9 \leq 2f(x) - 3 \leq 7$. Therefore, the range of $2f(x) - 3$ is $-9 \leq y \leq 7$.

8.8e Problem: Can you find constants B and C so that the domain of $f(B(x - C))$ is $8 \leq x \leq 9$?

8.8e Solution: When we put $B(x - C)$ into f, we are changing the input; therefore we can change the domain, or the valid inputs of x. The domain of $f(x)$ is $1 \leq x \leq 6$. That is, the lower bound is 1 and the upper bound is 6. We want to transform them such that the initial lower bound is 8 and the upper bound is 9, through the transformation $B(x - C)$. Therefore,

$$B(8 - C) = 1$$

$$B(9 - C) = 6$$

Solving:

$$\begin{aligned} 6B(8 - C) &= B(9 - C) \\ 6(8 - C) &= 9 - C \\ 48 - 6C &= 9 - C \\ 39 &= 5C \\ C &= \frac{39}{5} \\ B\left(\frac{45}{5} - \frac{39}{5}\right) &= 6 \\ \frac{6}{5}B &= 6 \\ B &= 5 \end{aligned}$$

Therefore, $B = 5$ and $C = \frac{39}{5}$.

8.8f Problem: Can you find constants A and D so that the range of $Af(x) + D$ is $0 \leq y \leq 1$?

8.8f Solution: The range of $f(x)$ is $-3 \leq f(x) \leq 5$. The lower bound is -3 and the upper bound is 5. We would like for the initial lower bound to be 0 and the upper bound to be 1 through $Af(x) + D$. Therefore,

$$A(-3) + D = 0$$

$$A(5) + D = 1$$

Subtracting these two yields

$$8A = 1$$

Therefore, $A = \frac{1}{8}$. Using this to find D:

$$\frac{5}{8} + D = 1 \rightarrow D = \frac{3}{8}$$

Hence, $A = \frac{1}{8}$ and $D = \frac{3}{8}$.

9 Question 8.9

8.9 Context: For each of the given functions $y = f(x)$, simplify the following expression so that h is no longer a factor in the denominator, then calculate the result of setting $h = 0$ in this simplified expression:

$$\frac{f(x + h) - f(x)}{h}$$

8.9a Problem: $f(x) = \frac{1}{x-1}$

8.9a Solution:

$$\frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} = \frac{1}{h(x+h-1)} - \frac{1}{h(x-1)} = \frac{x-1-(h+x-1)}{h\,(x-1)\,(h+x-1)} = -\frac{h}{h\,(x-1)\,(h+x-1)} = -\frac{1}{(x-1)\,(h+x-1)}$$

Plugging in $h = 0$ yields

$$-\frac{1}{(x-1)\,(0+x-1)} = -\frac{1}{(x-1)\,(x-1)} = -\frac{1}{(x-1)^2}$$

8.9b Problem: $f(x) = (2x+1)^2$

8.9b Solution:

$$\frac{(2(x+h)+1)^2 - (2x+1)^2}{h} = \frac{8xh + 4x^2 + 4x + 4h^2 + 4h + 1 - 4x^2 - 4x - 1}{h} = \frac{4h^2 + 8xh + 4h}{h} = 4h + 8x + 4$$

Plugging in $h = 0$ yields

$$0 + 8x + 4 = 8x + 4$$

8.9c Problem: $f(x) = \sqrt{25 - x^2}$

8.9c Solution:

$$\frac{\sqrt{25-(x+h)^2} - \sqrt{25-x^2}}{h} = \frac{\sqrt{25-(x+h)^2} - \sqrt{25-x^2}}{h} \cdot \frac{\sqrt{25-(x+h)^2} + \sqrt{25-x^2}}{\sqrt{25-(x+h)^2} + \sqrt{25-x^2}}$$

$$= \frac{(25-(x+h)^2) - (25-x^2)}{h\left(\sqrt{25-(x+h)^2} + \sqrt{25-x^2}\right)} = \frac{-h^2 - 2xh}{h\left(\sqrt{25-(x+h)^2} + \sqrt{25-x^2}\right)} = \frac{-h-2x}{\sqrt{25-(x+h)^2} + \sqrt{25-x^2}}$$

Plugging in $h = 0$ yields

$$\frac{-h-2x}{\sqrt{25-(x+h)^2} + \sqrt{25-x^2}} = \frac{-2x}{2\left(\sqrt{25-x^2}\right)} = \frac{-x}{\sqrt{25-x^2}}$$