Collingwood 38

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7.18 Problem: Consider the equation: $\alpha x^2 + 2\alpha^2 x + 1 = 0$. Find the values of x that make this equation true (your answer will involve α). Find the values of α that make this equation true (your answer will involve x).

7.18 Solution: Let us use the quadratic equation to solve for x.

$$x = \frac{-2\alpha^2 \pm \sqrt{(2\alpha^2)^2 - 4(\alpha)(1)}}{2(\alpha)} = \frac{-2\alpha^2 \pm \sqrt{4\alpha^4 - 4\alpha}}{2\alpha}$$

Therefore, the value of x that makes the equation true is $x = \frac{-2\alpha^2 \pm \sqrt{4\alpha^4 - 4\alpha}}{2\alpha}$.
Let us use the quadratic equation to solve for α. We will need to rewrite the equation as $2x\alpha^2 + x^2\alpha + 1 = 0$.

$$\alpha = \frac{-x^2 \pm \sqrt{(x^2)^2 - 4(2x)(1)}}{2(2x)} = \frac{-x^2 \pm \sqrt{x^4 - 8x}}{4x}$$

Therefore, the value of α that makes the equation true is $\alpha = \frac{-x^2 \pm \sqrt{x^4 - 8x}}{4x}$.

7.19 Context: For each of the following equations, find the value(s) of the constant α so that the equation has exactly one solution, and determine the solution for each value.

7.19a Problem: $\alpha x^2 + x + 1 = 0$

7.19a Solution: The discriminant must be 0 if the equation has exactly one solution. The equation is:

$$\begin{aligned} 1^2 - 4(\alpha)(1) &= 0 \\ 1 &= 4\alpha \\ \alpha &= \frac{1}{4} \end{aligned}$$

Therefore, $\alpha = \frac{1}{4}$. Solving for the solution of this:

$$\begin{aligned} \frac{1}{4}x^2 + x + 1 &= 0 \\ x^2 + 4x + 4 &= 0 \\ (x+2)^2 &= 0 \\ x &= -2 \end{aligned}$$

The solution in this case is $x = -2$.

7.19b Problem: $x^2 + \alpha x + 1 = 0$

7.19b Solution: The discriminant must be 0 if the equation has exactly one solution. The equation is:

$$\alpha^2 - 4(1)(1) = 0$$
$$\alpha^2 = 4$$
$$\alpha = \pm 2$$

Therefore, $\alpha = \pm 2$. Solving for one solution of this with $\alpha = 2$:

$$x^2 + 2x + 1 = 0$$
$$(x+1)^2 = 0$$
$$x = -1$$

Solving for another solution with $\alpha = -2$:

$$x^2 - 2x + 1 = 0$$
$$(x-1)^2 = 0$$
$$x = 1$$

The solutions in this case are $x = 1$ and $x = -1$.

7.19c Problem: $x^2 + x + \alpha = 0$

7.19c Solution: The discriminant must be 0 if the equation has exactly one solution. The equation is:

$$1^2 - 4(1)(\alpha) = 0$$
$$1 = 4\alpha$$
$$\alpha = \frac{1}{4}$$

Therefore, $\alpha = \frac{1}{4}$. Solving for one solution of this with $\alpha = 2$:

$$x^2 + x + \frac{1}{4} = 0$$
$$4x^2 + 4x + 1 = 0$$
$$(2x+1)^2 = 0$$
$$2x = -1$$
$$x = -\frac{1}{2}$$

The solution in this case is $x = -\frac{1}{2}$.

7.19d Problem: $x^2 + \alpha x + 4\alpha + 1 = 0$

7.19d Solution: The discriminant must be 0 if the equation has exactly one solution. The equation is:

$$\alpha^2 - 4(1)(4\alpha + 1) = 0$$
$$\alpha^2 - 16\alpha - 4 = 0$$
$$\alpha^2 - 16\alpha + 64 = 68$$
$$(\alpha - 8)^2 = 68$$
$$\alpha - 8 = \pm 2\sqrt{17}$$
$$\alpha = \pm 2\sqrt{17} + 8$$

Therefore, $\alpha = \pm 2\sqrt{17} + 8$. Solving for one solution of this with $\alpha = 2\sqrt{17} + 8$ with the quadratic formula:

$$\begin{aligned}
x^2 + (2\sqrt{17} + 8)x + 4(2\sqrt{17} + 8) + 1 &= 0 \\
x^2 + (2\sqrt{17} + 8)x + 8\sqrt{17} + 33 &= 0 \\
x &= \frac{-(2\sqrt{17} + 8) \pm \sqrt{0}}{2(1)} \\
&= -\frac{2\sqrt{17} + 8}{2} \\
&= -\sqrt{17} - 4
\end{aligned}$$

Note that we can assume the part of the quadratic formula, $b^2 - 4ac$, is equal to 0, because we found values of α in which that is true.

Solving for another solution with $\alpha = -2\sqrt{17} + 8$:

$$\begin{aligned}
x^2 + (-2\sqrt{17} + 8)x + 4(2\sqrt{17} + 8) + 1 &= 0 \\
x^2 + (-2\sqrt{17} + 8)x + 8\sqrt{17} + 33 &= 0 \\
x &= \frac{-(-2\sqrt{17} + 8) \pm \sqrt{0}}{2(1)} \\
&= -\frac{-2\sqrt{17} + 8}{2} \\
&= \sqrt{17} - 4
\end{aligned}$$

The solutions in this case are $x = -\sqrt{17} - 4$ and $x = \sqrt{17} - 4$.