# Collingwood 36

Andre Ye

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**7.14 Problem:** Steve likes to entertain friends at parties with "wire tricks." Suppose he takes a piece of wire 60 inches long and cuts it into two pieces. Steve takes the first piece of wire and bends it into the shape of a perfect circle. He then proceeds to bend the second piece of wire into the shape of a perfect square. Where should Steve cut the wire so that the total area of the circle and square combined is as small as possible? What is this minimal area? What should Steve do if he wants the combined area to be as large as possible?

**7.14 Solution:** Let $x$ be the length of one of the pieces of wire. It follow, then, that the other has length $60 - x$. If a circle is made with a piece of wire of length $x$, we can write that $2\pi r = x \to r = \frac{x}{2\pi}$. The area, then, is $\pi r^2 = \pi \left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$. On the other hand, if a square is made with a piece of length $60 - x$, it side length is $\frac{60-x}{4}$. The area is $\frac{(60-x)^2}{16} = \frac{x^2-120x+3600}{16} \to \frac{x^2}{16} - \frac{15x}{2} + 225$. Adding the two areas yields $\frac{x^2}{4\pi} + \frac{x^2}{16} - \frac{15x}{2} + 225$. Simplifying:

$$
\begin{aligned}
&\frac{x^2}{4\pi} + \frac{x^2}{16} - \frac{15x}{2} + 225 \\
&= \left(\frac{1}{4\pi} + \frac{1}{16}\right) x^2 - \frac{15}{2} x + 225 \\
&= \left(\frac{4}{16\pi} + \frac{\pi}{16\pi}\right) - \frac{15}{2} x + 225 \\
&= \left(\frac{4+\pi}{16\pi}\right) x^2 - \frac{15}{2} x + 225
\end{aligned}
$$

The vertex is at $-\frac{b}{2a} = -\frac{-\frac{15}{2}}{2\left(\frac{4+\pi}{16\pi}\right)} = -\left(-\frac{15}{4 \cdot \frac{\pi+4}{16\pi}}\right) = \frac{15 \cdot 4\pi}{4+\pi} = \frac{60\pi}{4+\pi}$. Therefore, the minimum area (the vertex is the minimum because the coefficient for the vertex is larger than 0) occurs when:

- $\boxed{\text{the circle is cut with } \frac{60\pi}{4+\pi} \approx 26.394 \text{ in.}}$
- $\boxed{\text{the square is cut with } 60 - \frac{60\pi}{4+\pi} \approx 33.606 \text{ in.}}$

The minimum area can be found by plugging in $x = \frac{60\pi}{4+\pi}$, which yields:

$$
\begin{aligned}
&\left(\frac{4+\pi}{16\pi}\right)\left(\frac{60\pi}{4+\pi}\right)^2 - \frac{15}{2}\left(\frac{60\pi}{4+\pi}\right) + 225 \\
&\approx 126.02231
\end{aligned}
$$

The $\boxed{\text{minimum area is about 126.02 inches squared}}$.

Because the parabola faces up, we need to find whether the area is larger at $x = 0$ or $x = 60$, the two extremes of the domain. The quadratic is symmetric about its vertex, therefore the value of $x$ that is farther from the vertex has a larger area. $x = 60$ is the farther one; therefore for the maximum area, $\boxed{\text{Steve should use all 60 inches to make a circle.}}$

**7.14 Context:** Two particles are moving in the $xy$-plane. The move along straight lines at constant speed. At time $t$, particle A's position is given by

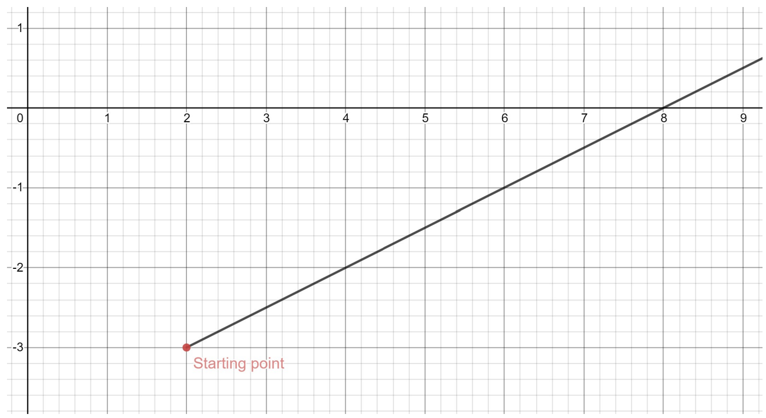
$$x = t + 2, y = \frac{1}{2}t - 3$$

and particle B's position is given by

$$x = 12 - 2t, y = 6 - \frac{1}{3}t.$$

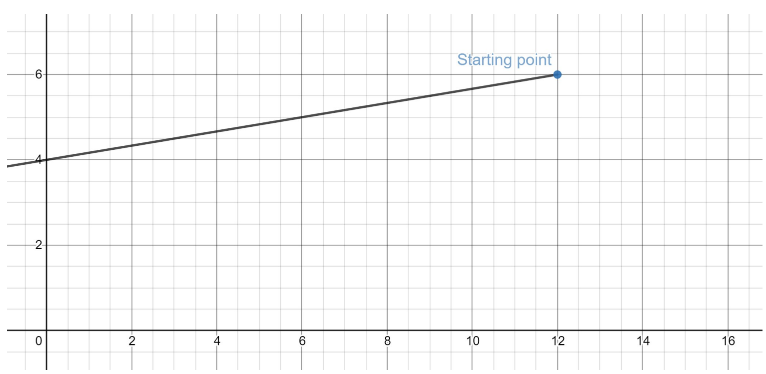
**7.14a Problem:** Find the equation of the line along which particle A moves. Sketch this line, and label A's starting point and direction of motion.

**7.14a Solution:** At any time $t$, the $x$-component is $t + 2$ and the $y$-component is $\frac{1}{2}t - 3$. Let us use $t = 0$ and $t = 2$ to produce two points on this line, which yield the points $(2, -3)$ and $(4, -2)$. The slope is $\frac{-2+3}{4-2} = \frac{1}{2}$. Therefore, the equation for the line is $y + 3 = \frac{1}{2}(x - 2) \rightarrow \boxed{y = \frac{1}{2}x - 4}$. The starting point is $t = 0$, which we have calculated to be at position $(2, -3)$.




**7.14b Problem:** Find the equation of the line along which particle B moves. Sketch this line on the same axes, and label B's starting point and direction of motion.

**7.14b Solution:** At any time $t$, the $x$-component is $12 - 2t$ and the $y$-component is $6 - \frac{1}{3}t$. Let us use $t = 0$ and $t = 3$ to produce two points on this line, which yields the points $(12, 6)$ and $(6, 5)$. The slope for this is $\frac{6-5}{12-6} = \frac{1}{6}$. Therefore, the equation for the line is $y - 5 = \frac{1}{6}(x - 6) \rightarrow \boxed{y = \frac{1}{6}x + 4}$. The starting point is $t = 0$, which we have calculated to be at position $(12, 6)$.




**7.14c Problem:** Find the time (i.e., the value of t) at which the distance between A and B is minimal. Find the locations of particles A and B at this time, and label them on your graph.

**7.14c Solution:** The distance between the two points, using the distance formula, is:

$$\begin{aligned}
&\sqrt{((t+2)-(12-2t))^2+\left(\left(\frac{1}{2}t-3\right)-\left(6-\frac{1}{3}t\right)\right)^2} \\
&= \sqrt{(3t-10)^2+\left(\frac{5t}{6}-9\right)^2} \\
&= \sqrt{\frac{349t^2-2700t}{36}+181} \\
&= \sqrt{\frac{349t^2-2700t+6516}{36}} \\
&= \sqrt{\frac{349t^2}{36}-75t+181}
\end{aligned}$$

The minimum of $\sqrt{\frac{349t^2}{36}-75t+181}$ is the minimum of $\frac{349}{36}t^2-75t+181$. The vertex of this is:

$$-\frac{-75}{2\left(\frac{349}{36}\right)}=\frac{75}{2\cdot\frac{349}{36}}=\frac{1350}{349}$$

Therefore, the two particles are the closest at $\boxed{t=\frac{1350}{349}.}$