Collingwood 34

Andre Ye

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7.9 Problem: Sylvia has an apple orchard. One season, her 100 trees yielded 140 apples per tree. She wants to increase her production by adding more trees to the orchard. However, she knows that for every 10 additional trees she plants, she will lose 4 apples per tree (i.e., the yield per tree will decrease by 4 apples). How many trees should she have in the orchard to maximize her production of apples?

7.9 Solution: Let t be the number of trees she has, and let a be the total number of apples she has. For every 10 additional trees she plants, she will lose 4 apples per tree; therefore, the total number of apples can be represented by $a = (100+t)(140 - \frac{4}{10}t)$ (for every additional tree added, $\frac{4}{10}$ apples are removed per tree). Expanding yields $a = -\frac{2}{5}t^2 + 100t + 14000$; the vertex is at $-\frac{b}{2a} = -\frac{100}{2(-\frac{2}{5})} = 125$. Therefore, she should have $100 + 125 = \boxed{225 \text{ trees}}$.

7.10 Problem: Rosalie is organizing a circus performance to raise money for a charity. She is trying to decide how much to charge for tickets. From past experience, she knows that the number of people who will attend is a linear function of the price per ticket. If she charges 5 dollars, 1200 people will attend. If she charges 7 dollars, 970 people will attend. How much should she charge per ticket to make the most money?

7.10 Solution: Let t be the price of each ticket, and let p represent the number of people who will attend. Given the two points, we can write the line representing the number of people present given the price as $p - 970 = \frac{1200-970}{5-7}(t-7) \to p - 970 = -115t + 805 \to p = -115t + 1775$. The total profit she gains is $p \cdot t$, which can be rewritten as $(-115t + 1775)t = -115t^2 + 1775t$. The vertex is $-\frac{b}{2a} = -\frac{1775}{2(-115)} = \frac{355}{46} \approx 7.72$. Therefore, Rosalie should charge $\boxed{\$7.72}$ to make the most money.

7.11 Problem: A Norman window is a rectangle with a semicircle on top. Suppose that the perimeter of a particular Norman window is to be 24 feet. What should its dimensions be in order to maximize the area of the window and, therefore, allow in as much light as possible?

7.11 Solution: Let the width and the height of the rectangle be represented as w and h; therefore the radius is $\frac{w}{2}$. The perimeter can be calculated as:

$$\begin{aligned}\text{perimeter} &= 2 \cdot \text{height} + \text{width} + \frac{\text{circumference}}{2} \\ &= 2h + w + \frac{2\pi \frac{w}{2}}{2} \\ &= 2h + w + \frac{\pi w}{2}\end{aligned}$$

It must be true that $2h + w + \frac{\pi w}{2} = 24$. We can express h as the following:

$$
\begin{aligned}
2h + w + \frac{\pi w}{2} &= 24 \\
2h &= 24 - w - \frac{\pi w}{2} \\
h &= \frac{24 - w - \frac{\pi w}{2}}{2} \\
&= \frac{48 - 2w - \pi w}{4}
\end{aligned}
$$

The area can be represented as:

$$
\begin{aligned}
\text{area} &= \text{width} \cdot \text{height} + \frac{\text{area of circle}}{2} \\
&= w \cdot \left(\frac{48 - 2w - \pi w}{4} \right) + \frac{\pi \left(\frac{w}{2} \right)^2}{2} \\
&= \frac{(48 - 2w - \pi w)\, w \cdot\ 2 + w^2 \pi}{8} \\
&= \frac{96w - 4w^2 - \pi w^2}{8} \\
&= \frac{-4 - \pi}{8} w^2 + 12w
\end{aligned}
$$

Avoiding ugly fractions, we can take the derivative and solve for the maximum:

$$
\begin{aligned}
\frac{-4 - \pi}{4} w + 12 &= 0 \\
\frac{-4 - \pi}{4} w &= -12 \\
(-4 - \pi) &= -48 \\
w &= \frac{-48}{-4 - \pi} = \frac{48}{4 + \pi}
\end{aligned}
$$

Therefore, the radius is $\frac{\frac{48}{4+\pi}}{2} = \frac{24}{4+\pi}$. The height, using the previous formula $h = \frac{48-2w-\pi w}{4} = \frac{48-2\left(\frac{48}{4+\pi}\right)-\pi\left(\frac{48}{4+\pi}\right)}{4} = \frac{48-\frac{96}{\pi+4}-\frac{48\pi}{\pi+4}}{4} = \frac{\frac{96}{4+\pi}}{4} = \frac{24}{4+\pi}$.

The dimensions are $\boxed{\text{height and radius of } \frac{24}{4+\pi} \text{ feet, width of } \frac{48}{4+\pi} \text{ feet}}$.