Collingwood 30

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7.1 Context: Write the following quadratic functions in vertex form, find the vertex, the axis of symmetry and sketch a rough graph.

7.1a Problem: $f(x) = 2x^2 - 16x + 41$.

7.1a Solution: Rewriting in vertex form:

$$\begin{aligned} f(x) &= 2x^2 - 16x + 41 \\ &= 2(x^2 - 8x + 16) + 41 - 32 \\ &= 2(x-4)^2 + 9 \end{aligned}$$

From vertex form, we see that the vertex is $(4, 9)$, from which it can be derived that the axis of symmetry is $x = 4$. Therefore, the vertex form is $\boxed{2(x-4)^2 - 9}$, the axis of symmetry is $\boxed{x = 4}$, and the vertex is $\boxed{(4, 9)}$.

7.1b Problem: $f(x) = 3x^2 - 15x - 77$.

7.1b Solution: Rewriting in vertex form:

$$\begin{aligned} f(x) &= 3x^2 - 15x - 77 \\ &= 3(x^2 - 5x + \frac{25}{4}) - 77 - \frac{75}{4} \\ &= 3(x^2 - \frac{5}{2})^2 - \frac{383}{4} \end{aligned}$$

From vertex form, we see that the vertex is $(\frac{5}{2}, -\frac{383}{4})$, from which it can be derived that the axis of symmetry is $x = \frac{5}{2}$. Therefore, the vertex form is $\boxed{3\left(x^2 - \frac{5}{2}\right)^2 - \frac{383}{4}}$, the axis of symmetry is $\boxed{x = \frac{5}{2}}$, and the vertex is $\boxed{\left(\frac{5}{2}, -\frac{383}{4}\right)}$.

7.1c Problem: $f(x) = x^2 - \frac{3}{7}x + 13$.

7.1c Solution: Rewriting in vertex form:

$$\begin{aligned} f(x) &= x^2 - \frac{3}{7}x + 13 \\ &= x^2 - \frac{3}{7}x + \frac{9}{196} + 13 - \frac{9}{196} \\ &= \left(x - \frac{3}{14}\right)^2 + \frac{2539}{196} \end{aligned}$$

From vertex form, we see that the vertex is $\left(\frac{3}{14}, \frac{2539}{196}\right)$, from which it can be derived that the axis of symmetry is $x = \frac{3}{14}$. Therefore, the vertex form is $\boxed{\left(x - \frac{3}{14}\right)^2 + \frac{2539}{196}}$, the axis of symmetry is $\boxed{x = \frac{3}{14}}$, and the vertex is $\boxed{\left(\frac{3}{14}, \frac{2539}{196}\right)}$.

7.1d Problem: $f(x) = 2x^2$.

7.1d Solution: Rewriting in vertex form:

$$\begin{aligned} f(x) &= 2x^2 \\ &= 2(x-0)^2 + 0 \end{aligned}$$

From vertex form, we see that the vertex is $(0, 0)$, from which it can be derived that the axis of symmetry is $x = 0$. Therefore, the vertex form is $\boxed{2(x-0)^2 + 0}$, the axis of symmetry is $\boxed{x = 0}$, and the vertex is $\boxed{(0, 0)}$.

7.1e Problem: $f(x) = \frac{1}{100}x^2$.

7.1e Solution: Rewriting in vertex form:

$$\begin{aligned} f(x) &= \frac{1}{100}x^2 \\ &= \frac{1}{100}(x-0)^2 + 0 \end{aligned}$$

From vertex form, we see that the vertex is $(0, 0)$, from which it can be derived that the axis of symmetry is $x = 0$. Therefore, the vertex form is $\boxed{\frac{1}{100}(x-0)^2 + 0}$, the axis of symmetry is $\boxed{x = 0}$, and the vertex is $\boxed{(0, 0)}$.

Figure 1: Red: $2x^2 - 16x + 41$, Blue: $3x^2 - 15x - 77$, Green: $x^2 - \frac{3}{7}x + 13$, Purple: $2x^2$, Black: $\frac{1}{100}x^2$.

7.2 Context: In each case, find a quadratic function whose graph passes through the given points:

7.2a Problem: $(0, 0), (1, 1)$ and $(3, -1)$.

7.2a Solution: Consider coefficients a, b, and c in the quadratic equation $ax^2 + bx + c$. We have three systems of equations:

$$\begin{cases} a0 + b0 + c = 0 \to c = 0 \\ a1 + b1 + c = 1 \to a + b + c = 1 \\ a9 + b3 + c = -1 \to 9a + 3b + c = -1 \end{cases}$$

We know that $c = 0$, so we can simplify the other two equations as $a + b = 1$ and $9a + 3b = -1$. We can substitute b in the latter equation with $b = 1 - a$ (derived from the first equation) to form $9a + 3(1 - a) = -1 \to 9a + 3 - 3a = -1 \to 6a = -4 \to a = -\frac{2}{3}$. Then, b is equal to $1 - a = \frac{5}{3}$. Therefore, the quadratic is $\boxed{y = -\frac{2}{3}x^2 + \frac{5}{3}x}$.

7.2b Problem: $(-1, 1), (1, -2)$ and $(3, 4)$.

7.2b Solution: Consider coefficients a, b, and c in the quadratic equation $ax^2 + bx + c$. We have three systems of equations:

$$\begin{cases} a(1) + b(-1) + c = 1 \to a - b + c = 1 \\ a1 + b1 + c = -2 \to a + b + c = -2 \\ a9 + b3 + c = 4 \to 9a + 3b + c = 4 \end{cases}$$

We can subtract the first equation from the second to produce $2b = -3 \to b = -\frac{3}{2}$. Therefore, we have that $a - \frac{3}{2} + c = -2 \to a + c = -\frac{1}{2}$ and $9a + 3\left(-\frac{3}{2}\right) + c = 4 \to 9a + c = \frac{17}{2}$. Subtracting this from the first derived equation yields $8a = \frac{18}{2} = 9 \to a = \frac{9}{8}$. Plugging this into $a + c = -\frac{1}{2}$, it is apparent that $-\frac{13}{8}$. Therefore, the quadratic is $\boxed{y = \frac{9}{8}x^2 - \frac{3}{2}x - \frac{13}{8}}$.

7.2c Problem: $(2, 1), (3, 2)$ and $(5, 1)$.

7.2c Solution: Consider coefficients a, b, and c in the quadratic equation $ax^2 + bx + c$. We have three systems of equations:

$$\begin{cases} a4 + b2 + c = 1 \to 4a + 2b + c = 1 \\ a9 + b3 + c = 2 \to 9a + 3b + c = 2 \\ a25 + b5 + c = 1 \to 25a + 5b + c = 4 \end{cases}$$

Subtracting the second from the first yields $5a + b = 1$, and subtracting the third from the second yields $16a + 2b = -1$. Multiplying the first derived equation yields $10a + 2b = 2$. Subtracting this from the second derived equation yields $6a = -3$, hence $a = -\frac{1}{2}$. Thus, $5\left(-\frac{1}{2}\right) + b = 1 \to -\frac{5}{2} + b = 1 \to b = \frac{7}{2}$. Therefore, $4\left(-\frac{1}{2}\right) + 2\left(\frac{7}{2}\right) + c = 1 \to -\frac{4}{2} + \frac{14}{2} + c = 1 \to -2 + 7 + c = 1 \to c = -4$. Therefore, the quadratic is $\boxed{y = -\frac{1}{2}x^2 + \frac{7}{2}x - 4.}$

7.2d Problem: $(0, 1), (1, 1)$ and $(1, 3)$.

7.2d Solution: Consider coefficients a, b, and c in the quadratic equation $ax^2 + bx + c$. We have three systems of equations:

$$\begin{cases} a0 + b0 + c = 1 \to c = 1 \\ a1 + b1 + c = 1 \to a + b + c = 1 \\ a1 + b1 + 1 = 9 \to a + b + c = 9 \end{cases}$$

Setting the second and third equations equal to each other yields $1 = 9$, which is not true. Therefore, there are $\boxed{\text{no solutions.}}$