Collingwood 28

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6.11 Context: The vertical cross-section of a drainage ditch is pictured. Here, R indicates a circle of radius 10 feet and all of the indicated circle centers lie along the common horizontal line 10 feet above and parallel to the ditch bottom. Assume that water is flowing into the ditch so that the level above the bottom is rising 2 inches per minute.

6.11a Problem: When will the ditch be completely full?

6.11a Solution: The water flows into the ditch such that the level above the bottom rises at 2 inches per minute. Meanwhile, the height of the ditch is $10 \cdot 2 = 20$ feet high, or $20 \cdot 12 = 240$ inches. Hence, it takes $\frac{240}{2} = 120$ minutes, or $\boxed{2 \text{ hours}}$, for the ditch to be completely full.

6.11b Problem: Find a multipart function that models the vertical cross-section of the ditch.

6.11b Solution: Let the bottom of the ditch be represented with the x-axis, and let the vertical line that cuts the ditch in half in the picture represent the y-axis. Let the four circles be referred to as A, B, C, and D, in the order that they are pictured from left to right. There are multiple components that need to e accounted for.

- When $x < -40$, the ground is flat at $y = 20$.
- From $-40 \leq x < -30$, the ditch is modelled by the top-right quarter of circle A, which is centered at $(-40, 10)$ and has radius 10, and thus can be represented as $(x+40)^2 + (y-10)^2 = 100$. Rewriting this in terms of y, the arc is modelled by $y = 10 + \sqrt{100 - (x+40)^2}$.
- From $-30 \leq x < -20$, the ditch is modelled by the bottom-left quarter of circle B, which is centered at $(-20, 10)$ and has radius 10, and thus can be represented as $(x+20)^2 + (y-10)^2 = 100$. Rewriting this in terms of y, the arc is modelled by $y = 10 - \sqrt{100 - (x+20)^2}$. Note that the result of the square root is negative in this case because we are looking for solutions that fall below $y = 10$, whereas in the previous case we were looking for solutions that fell above $y = 10$.
- From $-20 \leq x < 20$, the ditch is flat and can be modelled by $y = 0$.
- From $20 \leq x < 30$, the ditch is modelled by the bottom-right quarter of circle C, which is centered at $(20, 10)$ and has radius 10. From the diagram, it is clear that this is the reflection of the portion from $-30 \leq x < -20$, which had equation $y = 10 - \sqrt{100 - (x+20)^2}$. However, because the x-location changed from $x = -20$ to $x = 20$, the equation changes to $y = 10 - \sqrt{100 - (x-20)^2}$.
- From $30 \leq x < 40$, the ditch is modelled by the top-left quarter of circle D, which is centered at $(40, 10)$ and has radius 10. From the diagram, it is clear that this is the reflection of the portion from $-40 \leq x < -30$. However, because the x-location changed from $x = -40$ to $x = 40$, the equation changes to $y = 10 + \sqrt{100 - (x-40)^2}$.
- From $x \leq 40$, the ground is flat at $y = 20$.

These components can be assembled into a multi-part function:

$$y = \begin{cases} 20 & \text{if } x < -40 \\ 10 + \sqrt{100 - (x+40)^2} & \text{if } -40 \leq x < -30 \\ 10 - \sqrt{100 - (x+20)^2} & \text{if } -30 \leq x < -20 \\ 0 & \text{if } -20 \leq x < 20 \\ 10 - \sqrt{100 - (x-20)^2} & \text{if } 20 \leq x < 30 \\ 10 + \sqrt{100 - (x-40)^2} & \text{if } 30 \leq x < 40 \\ 20 & \text{if } x \geq 40 \end{cases}$$

6.11c Problem: What is the width of the filled portion of the ditch after 1 hour and 18 minutes?

6.11c Solution: 1 hour and 18 minutes can be converted into $60 + 18 = 78$ minutes, in which the water, rising at 2 inches per minute, has risen $78 \cdot 2 = 156$ inches, or 13 feet. In discussion of the four circles A, B, C, and D, only certain ranges of each circle are relevant. Circles A and D appear from range $10 \leq y \leq 20$ (using the same coordinate system imposed in 6.11b solution), whereas circles B and C appear from range $0 \leq y \leq 10$. Since we are looking for $y = 13$, we only need to concern ourselves with circles A and D. Furthermore, these arcs are reflections of each other across the y-axis, so one can simply find the x value of which $y = 13$ in circle D and double it to find the width.

Recall circle D is given by $y = 10 + \sqrt{100 - (x-40)^2}$.

$$\begin{aligned} 13 &= 10 - \sqrt{100 - (x-40)^2} \\ 9 &= 100 - (x-40)^2 \\ -x^2 + 80x - 1500 &= 9 \\ x^2 - 80x + 1509 &= 0 \\ x &= \frac{80 \pm \sqrt{(-80)^2 - 4(1)(1509)}}{2(1)} \\ &= 40 \pm \sqrt{91} \end{aligned}$$

$40+\sqrt{91}$ evaluates to about 49.5394. This exceeds the given range for this section, which was $30 \leq x < 40$. Therefore, the solution is $40 - \sqrt{91}$; doubling this as discussed above yields a width of $\boxed{2\left(40 - \sqrt{91}\right) \text{ feet.}}$

6.11d Problem: When will the filled portion of the ditch be 42 feet wide? 50 feet wide? 73 feet wide?

6.11d Solution: We can apply the same logic used to solve 6.11c, but reversed. When the filled portion of the ditch is 42 feet wide, the relevant values of x are $\pm\frac{42}{2} = \pm 21$. Looking at the multi-part function constructed by 6.11b, it becomes clear that this particular case is relevant for circles B and C. The equation of circle C is $y = 10 - \sqrt{100 - (x-20)^2}$; thus at $x = 21$, $y = 10 - \sqrt{100 - (21-20)^2} = 10 - 3\sqrt{11}$. Converting this from feet to inches yields $12 \cdot \left(10 - 3\sqrt{11}\right)$; the ditch fills at 2 inches per minute, so it takes $\frac{12 \cdot \left(10 - 3\sqrt{11}\right)}{2} \approx 0.30075$ minutes to be 42 feet wide.

Using the same steps, at 50 feet wide the relevant circles are B and C. Utilizing the equation of circle C, at $x = 25$, $y = 10 - \sqrt{100 - (25-20)^2} = 10 - 5\sqrt{3}$. Converting this from feet to inches yields $12 \cdot (10 - 5\sqrt{3})$; given the speed the ditch fills, it takes $\frac{12 \cdot (10 - 5\sqrt{3})}{2} \approx 8.038476$ minutes to be 50 feet wide.

At 73 feet wide, the relevant circles are A and D. Utilizing the equation of circle D, at $x = 36.5$, $y = 10 + \sqrt{100 - (36.5-40)^2} \approx 19.36749$. Converting this to inches yields $12 \cdot 19.36749 = 232.40988$; hence it takes $\frac{232.40988}{2} = 116.20494$ minutes to be 73 feet wide.

Thus, the solutions are:

- It takes ≈ 0.3008 minutes for the filled portion to be 42 feet wide.

- It takes ≈ 8.0385 minutes for the filled portion to be 50 feet wide.
- It takes ≈ 116.205 minutes for the filled portion to be 73 feet wide.

6.12 Context: The graph of a function $y = g(x)$ on the domain $-6 \leq x \leq 6$ consists of line segments and semicircles of radius 2 connecting the points $(-6,0), (-4,4), (0,4), (4,4), (6,0)$.

6.12a Problem: What is the range of g?

6.12a Solution: Looking at the graph, the minimum value of $g(x)$ is 0. The maximum is formed by the semicircle from $(-4,4)$ to $(0,4)$. This has a radius of 2; thus the maximum is $4+2=6$. Hence, the range is $\boxed{0 \leq g(x) \leq 6.}$

6.12b Problem: Where is the function increasing? Where is the function decreasing?

6.12b Solution: Looking at the graph, we can conclude:

Domain Section	Status
$-6 \leq x < -2$	Increasing
$x = -2$	Flat (not increasing or decreasing)
$-2 < x < 2$	Decreasing
$x = 2$	Flat (not increasing or decreasing)
$2 < x \leq 4$	Increasing
$4 < x \leq 6$	Decreasing

6.12c Problem: Find the multipart formula for $y = g(x)$.

6.12c Solution: Each of the components can be considered as such:

- From $-6 \leq x < -4$ is a line with slope 2 passing through the points $(-6,0)$ and $(-4,4)$; thus the line can be represented with $y = 2(x+6) = 2x+12$.
- From $-4 \leq x < 0$ is a semicircle centered at $(-2,4)$ with radius 2; thus it can be written as $(x+2)^2 + (y-4)^2 = 4$. However, this creates a circle and not a semicircle; we can rewrite the equation as $y = 4 + \sqrt{4-(x+2)^2}$ to include only the solutions that satisfy $y \geq 4$.
- From $0 \leq x < 4$ is another semicircle centered at $(2,4)$ with radius 2; thus it can be written as $(x-2)^2 + (y-4)^2 = 4$. However, this creates a circle and not a semicircle; we can rewrite the equation as $y = 4 - \sqrt{4-(x-2)^2}$ to include only the solutions that satisfy $y \leq 4$.
- From $4 \leq x \leq 6$ is a line that is the reflection of the other line discussed initially; therefore its equation is $-2x+12$ (the slope is negated and the y-intercept remains the same).

Combining these, the multipart formula is:

$$y = \begin{cases} 2x+12 & \text{if } -6 \leq x < -4 \\ 4+\sqrt{4-(x+2)^2} & \text{if } -4 \leq x < 0 \\ 4-\sqrt{4-(x-2)^2} & \text{if } -0 \leq x < 4 \\ -2x+12 & \text{if } 4 \leq x \leq 6 \end{cases}$$

6.12d Problem: If we restrict the function to the smaller domain $-5 \leq x \leq 0$, what is the range?

6.12d Solution: Using our multipart function, we can find the range with the minimum and maximum range value; from the graph, the minimum range can be found at the lower bound of the domain, $x = -5$, which is $y = 2(-5) + 12 = 2$ and $y = 4$, respectively. On the other hand, the maximum is the top of the semicircle, which is located at height $y = 6$ (from a circle whose origin has y-value 4 and radius 2). Therefore, the range is $\boxed{2 \leq y \leq 6.}$

6.12e Problem: If we restrict the function to the smaller domain $0 \leq x \leq 4$, what is the range?

6.12e Solution: The range $0 \leq x \leq 4$ is simply the range of the semicircle displayed in the right section of the diagram. The lowest point is $y = 2$ (from a circle whose origin has y-value 4 and radius 2), and the highest point is $y = 4$ (the two edges of semicircle). Therefore, the range is $\boxed{2 \leq y \leq 4}$.