Collingwood 25

Andre Ye

9 November 2020

6.5 Problem: Express the area of the shaded region below as a function of x. The dimensions in the figure are centimeters.

6.5 Solution: Let a coordinate system be imposed onto the trapezoid such that its bottom left point is at $(0,0)$. Therefore, the top left point is located at $(0,3)$, the bottom right point is located at $(6,0)$, and the top point is located at $(6,5)$. The line at the top of the diagram can be represented, then, with equation $y = \frac{1}{3}x + 3$. Consider the re-sizable trapezoid from a 90-degree tilted angle, in which the diagram is a trapezoid with top base length of 3, bottom base length of $y = \frac{1}{3}x + 3$, and a height of x. The maximum height of the trapezoid is 6 units.

The area for a trapezoid is given by the sum of the two bases multiplied by the height, divided by two. This is $\frac{(3+\frac{1}{3}x+3)x}{2} = \frac{(6+\frac{1}{3}x)x}{2} = \frac{\frac{1}{3}x^2+6x}{2} = \frac{1}{6}x^2 + \frac{1}{3}$. Thus, the area of the shaded region is $\boxed{\frac{1}{6}x^2 + \frac{1}{3}\text{cm}^2}$, with domain $0 \leq x \leq 6$.

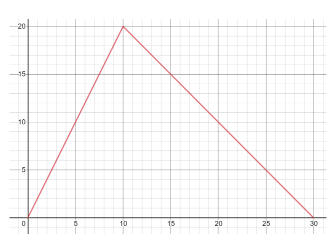
6.6 Context: Pizzeria Buonapetito makes a triangular-shaped pizza with base width of 30 inches and height 20 inches as shown. Alice wants only a portion of the pizza and does so by making a vertical cut through the pizza and taking the shaded portion. Letting x be the bottom length of Alice's portion and y be the length of the cut as shown, answer the following questions:

6.6a Problem: Find a formula for y as a multipart function of x, for $0 \leq x \leq 30$. Sketch the graph of this function and calculate the range.

6.6a Solution: We must treat the triangle in two parts: a right triangle for $x \leq 10$, which will be referred to as triangle A, and another for $x > 10$, which will be referred to as triangle B. The hypotenuse of A can be found by imposing it on a coordinate system; let the bottom left vertex be $(0,0)$ and the top right vertex be $(10,20)$. Thus, the line representing this hypotenuse is $y = 2(x-10)+20$. Similarly, the hypotenuse of B can be found by imposing it on a coordinate system; let the top left vertex be $(10,20)$ and the bottom right vertex be $(30,0)$. Thus, the line representing this hypotenuse is $y = -(x-30) = -x+30$. Our multi-part function is then:

$$y = \begin{cases} 2(x-10)+20 & \text{if } 0 \leq x \leq 10 \\ -x+30 & \text{if } 10 < x < 30 \end{cases} \tag{1}$$

The minimum value of y is 0, and the maximum is 20 (at its peak). Thus, the range of y is $\boxed{0 \leq y \leq 20}$.



6.6b Problem: Find a formula for the area of Alice's portion as a multi-part function of x, for $0 \leq x \leq 30$.

6.6b Solution: We have already defined the height of the triangle at a certain value of x for sub-triangles A and B. If $x \leq 10$, the area is simply the fraction of triangle A that has been cut. As the height of the triangle is given by $2(x-10)+20 = 2x$, and the equation for the area of a triangle is $\frac{bh}{2}$, where b is the length of the base and h is the length of the height, the area comes out to be $\frac{x \cdot 2x}{2} = x^2$ units squared.

If $x > 10$, the area is the complete area of A plus the area of the trapezoid formed by the cut part of B. The maximum area of triangle A, at $x = 10$, is 100 units squared. The area of a trapezoid is $\frac{(20-x+30)(x-10)}{2} = \frac{-x^2+60x-500}{2} = -\frac{1}{2}x^2 + 30x - 250$. Adding this to the area of triangle A yields $-\frac{1}{2}x^2 + 30x - 150$.

Therefore, the multi-part formula of the area at x, $a(x)$, is:

$$a(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 10 \\ -\frac{1}{2}x^2 + 30x - 150 & \text{if } 10 < x < 30 \end{cases} \tag{2}$$

6.6c Problem: If Alice wants her portion to have half the area of the pizza, where should she make the cut?

6.6c Solution: The total area of the triangle is $\frac{30 \cdot 20}{2} = 300$ units squared. Thus, we need to find the value of x such that $a(x) = 150$. We need to solve for both equations:

$$\begin{aligned} x^2 &= 150 \\ x &= \sqrt{150} = 5\sqrt{6} \approx 12.247. \end{aligned}$$

This is an invalid solution, because it does not satisfy the requirement $0 \leq x \leq 10$.

$$\begin{aligned} -\frac{1}{2}x^2 + 30x - 150 &= 150 \\ -x^2 + 60x - 300 &= 300 \\ x^2 - 60x + 300 &= -300 \\ x^2 - 60x + 600 &= 0 \\ x &= \frac{60 \pm \sqrt{3600 - 4(1)(600)}}{2} \\ &= 30 \pm 10\sqrt{3} \approx 12.6795 \text{ or } 47.3205 \end{aligned}$$

The latter solution does not satisfy $10 < x \leq 30$; therefore the solution is $\boxed{x = 30 - 10\sqrt{3} \approx 12.6795}$.