Collingwood 17

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4.12 Context: The infamous crawling tractor sprinkler is located as pictured below, 100 feet South of a 10 ft. wide sidewalk; notice the hose and sidewalk are not perpendicular. Once the water is turned on, the sprinkler waters a circular disc of radius 20 feet and moves North along the hose at the rate of $\frac{1}{2}$ inch/second.

4.12a Problem: Impose a coordinate system. Describe the initial coordinates of the sprinkler and find the equation of the line forming the southern boundary of the sidewalk.

4.12a Solution: Let the intersection of the hose and the southern edge of the sidewalk be the origin $(0, 0)$. Then,

- The southern edge of the sidewalk has equation $y = -\frac{1}{5}x$.
- The moving tractor begins at location $(0, -100)$.
- The location of the moving tractor after t minutes is $(0, \frac{5}{2}t - 100)$
- The equation of the circle representing the circular watered zone is $x^2 + (\frac{5}{2}t - 100)^2 = 400$.

4.12b Problem: After 33 minutes, sketch a picture of the wet portion of the sidewalk; find the length of the wet portion of the Southern edge of the sidewalk.

4.12b Solution: After 33 minutes, utilizing the derived formula for location of the moving tractor, the tractor will be located at $(0, \frac{5}{2} \cdot 33 - 100) \to (0, -\frac{35}{2}$, or $(0, -17.5)$. The equation of the circle representing the watered area is thus $x^2 + (y + 17.5)^2 = 400$. To find the two points of the watered area of the Southern edge of the sidewalk, we can find values for which the two equations intersect.

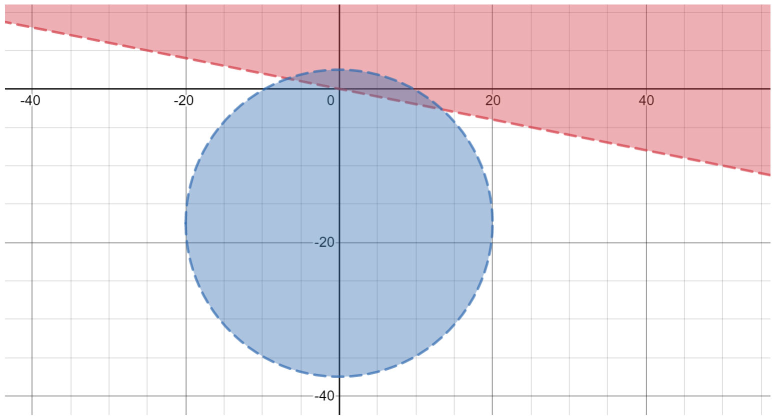
$$\begin{aligned}
x^2 + (-\frac{1}{5}x + \frac{35}{2})^2 &= 400 \\
x^2 + \frac{1}{25}x^2 - 7x + \frac{1225}{4} &= 400 \\
26x^2 - 175x + \frac{30,625}{4} &= 10,000 \\
104x^2 - 700x + 30,625 &= 40,000 \\
104x^2 - 700x - 9,375 &= 0
\end{aligned}$$

Solving for x:

$$\begin{aligned}
x &= \frac{700 \pm \sqrt{(-700)^2 - 4(104)(-9,375)}}{2(104)} \\
&= \frac{700 \pm \sqrt{4,390,000}}{208}
\end{aligned}$$

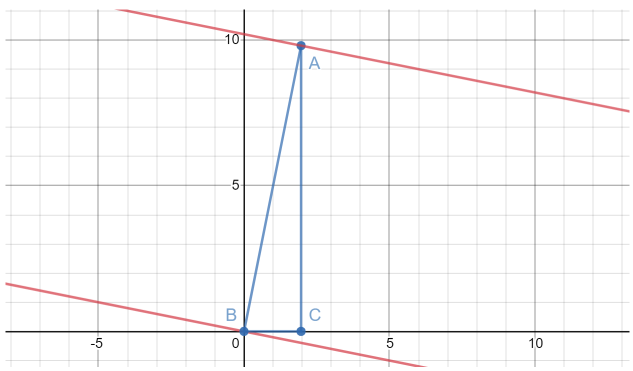
x comes out to be approximately 13.438619 or -6.707849. Plugging these into $y = -\frac{1}{5}x$ yields y-values of -2.6877 and 1.3416, respectively. Hence, the two points are $\boxed{(13.4386, -2.6877) \text{ and } (-6.7078, 1.3416)}$.

The distance between these two points is $\sqrt{(13.4386+6.7078)^2+(-2.6877-1.3416)^2} \approx 20.5454$. Hence, the length of the Southern edge of the sidewalk that is watered is about $\boxed{20.55 \text{ feet}}$.



4.12c Problem: Find the equation of the line forming the northern boundary of the sidewalk.

4.12c Solution: The northern boundary of the sidewalk is parallel to the southern boundary, it still has slope $-\frac{1}{5}$. We can draw a right triangle between these two lines and label points A, B, and C as in the image below. The distance from A to B is 10 feet, since the sidewalk is 10 feet wide. The line from point A to point B is also perpendicular to the North and South boundaries of the sidewalk. Thus, it has slope 5, and can be represented with the equation $y = 5x$ (its y-intercept is 0 since it passes through the origin).



Since the line $\overline{AB}$ has a slope of 5, the ratio of the length of line $\overline{AC}$ to the length of line $\overline{BC}$ is 5 : 1. Thus, we can represent their lengths as $5x$ and x, respectively. Using the Pythagorean Theorem, the following equation must be true:

$$\begin{aligned} 25x^2 + x^2 &= 10^2 \\ 26x^2 &= 100 \\ x &= \frac{10}{\sqrt{26}} \end{aligned}$$

Thus, the point A lies at location $(\frac{10}{\sqrt{26}}, \frac{50}{\sqrt{26}})$ (the y-coordinate was derived from the equation $y = 5x$). Since we know the North boundary of the sidewalk has slope $-\frac{1}{5}$, we can write its equation in point-slope form, we have $y - \frac{50}{\sqrt{26}} = -\frac{1}{5}(x - \frac{10}{\sqrt{26}})$. Distributing and simplifying, we get(approximately) $\boxed{y = -\frac{1}{5}x + 10.198}$.

4.13 Context: Margot is walking in a straight line from a point 30 feet due east of a statue in a park toward a point 24 feet due north of the statue. She walks at a constant speed of 4 feet per second.

4.13a Problem: Write parametric equations for Margot's position t seconds after she starts walking.

4.13a Solution: Let the statue be located at the origin $(0, 0)$. Thus, Margot begins at location $(30, 0)$ and begins walking towards point $(0, 24)$. Thus path has slope $-\frac{24}{30} = -\frac{4}{5}$. If we are to visualize a right triangle with vertices:

- Point A fixed at $(30, 0)$.
- Point B as Margot's current location. This ends as $(0, 24)$.
- Point C as Margot's x-location on the $x - axis$. This ends as $(0, 0)$.

...we can set AB - the hypotenuse of the right triangle - to be 4, the number of feet she walks in a second. Then, we can find the vertical and horizontal components of her movement.

- Line $\overline{AB}$ has length 4 feet.
- Line $\overline{AC}$ has length $5x$ feet.
- Line $\overline{BC}$ has length $4x$ feet.

By the Pythagorean Theorem, $25x^2 + 16x^2 = 16 \rightarrow 41x^2 = 16 \rightarrow x = \frac{4}{\sqrt{41}}$. Thus, at every second, Margot moves West $\frac{20}{\sqrt{41}}$ feet and North $\frac{16}{\sqrt{41}}$ feet. Since she begins at location $(30, 0)$, her location at time t (in seconds) becomes approximately $\boxed{(30 - 3.1235t, 2.4988t).}$

4.13b Problem: Write an expression for the distance from Margot's position to the statue at time t.

4.13b Solution: We know Margot's location at time t to be $(30 - 3.1235t, 2.4988t)$. Because the statue is centered at location (0,0), her distance from the statue at time t by the distance formula is $\sqrt{(30 - 3.1235t)^2 + (2.4988t)^2}$. Simplifying yields (about) $\boxed{\sqrt{16t^2 - 187.41t + 900}}$.

4.13c Problem: Find the times when Margot is 28 feet from the statue.

4.13c Solution: Using our derived formula, $\sqrt{16t^2 - 187.41t + 900}$, we can solve for the times t in which Margot is 28 feet away from the statue.

$$\sqrt{16t^2 - 187.41t + 900} = 28$$
$$16t^2 - 187.41t + 900 = 784$$
$$16t^2 - 187.41t + 116 = 0$$
$$t = \frac{187.41 \pm \sqrt{(-187.41)^2 - 4(16)(116)}}{2(16)}$$
$$t = \frac{187.41 \pm \sqrt{27698.5081}}{32}$$

Thus, there are two times for which Margot is 28 feet away from the status: $\boxed{t \approx 11.057 \text{ and } t \approx 0.656.}$