Collingwood 14

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4.3a Problem: What is the area of the triangle determined by the lines $y = -\frac{1}{2}x + 5$, $y = 6x$, and the y-axis?

4.3a Solution: We can find the points of intersection, then use the shoelace algorithm to find the area. Because one of the three lines is the y-axis, the other two lines have unique intersection points where $x = 0$.

$$\begin{aligned} y &= -\frac{1}{2}x + 5 \\ y &= -\frac{1}{2}0 + 5 \\ y &= 5 \end{aligned}$$

Thus, the intersection point for one of the lines is $(0, 5)$.

$$\begin{aligned} y &= 6x \\ y &= 0 \end{aligned}$$

Thus, the intersection point for one of the lines is $(0, 0)$. Another intersection point lies between the two non-y-axis lines:

$$\begin{aligned} -\frac{1}{2}x + 5 &= 6x \\ -\frac{13}{2}x &= -5 \\ x &= -5\left(-\frac{2}{13}\right) \\ x &= \frac{10}{13} \\ y = 6x &= \frac{60}{13} \end{aligned}$$

Thus, the intersection is $(\frac{10}{13}, \frac{60}{13})$. Writing out these points in shoelace format:

$$\begin{array}{c|c} 0 & 5 \\ 0 & 0 \\ \frac{10}{13} & \frac{60}{13} \\ 0 & 5 \end{array}$$

Cross-multiplying from top-right to bottom-left yields $\frac{50}{13}$. Cross-multiplying from top-left to bottom-right yields 0. Averaging the two yields $\frac{25}{13}$. Hence, the area is $\boxed{\frac{25}{13} \text{ square units.}}$

4.3b Problem: If $b > 0$ and $m < 0$, then the line $y = mx + b$ cuts off a triangle from the first quadrant. Express the area of that triangle in terms of m and b.

4.3b Solution: The one point given is $(0, b)$, or the y-intercept. Given this point, we can find a point where y equals 0, the x-intercept.

$$\begin{aligned} y &= mx + b \\ 0 &= mx + b \\ mx &= -b \\ x &= -\frac{b}{m} \end{aligned}$$

Thus, the x-intercept is located at $(-\frac{b}{m}, 0)$. The distance between these intercepts and the origin (0, 0 form the two legs of a right triangle. Since the y-intercept is $(0, b)$, the vertical leg has length b. Since the x-intercept is $(-\frac{b}{m}, 0)$, the horizontal leg has length $-\frac{b}{m}$. Using the area for triangle area:

$$\begin{aligned} \text{Area of Triangle} &= \frac{\text{Width} \cdot \text{Height}}{2} \\ &= \frac{-\frac{b}{m} \cdot b}{2} \\ &= \frac{-\frac{b^2}{m}}{2} \\ &= \frac{-b^2}{2m} \end{aligned}$$

Thus, the area of such a triangle is given by $\boxed{\frac{-b^2}{2m} \text{ square units}}$.

4.3c Problem: The lines $y = mx + 5$, $y = x$ and the y-axis form a triangle in the first quadrant. Suppose this triangle has an area of 10 square units. Find m.

4.3c Solution: The two fixed lines are $y = x$ and $x = 0$ (the y-axis). Their point of intersection is $(0, 0)$ (the origin). Each line has one point of intersection with $y = mx + 5$, which we can express in terms of m.

$$\begin{aligned} x &= mx + 5 \\ x - mx &= 5 \\ (1 - m)x &= 5 \\ x &= \frac{5}{1 - m} \end{aligned}$$

Thus, one intersection point is $(\frac{5}{1-m}, \frac{5}{1-m})$. The other intersection point between $y = mx + 5$ and the y-axis is simply the y-intercept, which remains 5 regardless of the value of m. Thus, the three intersections are $(0, 0)$, $(\frac{5}{1-m}, \frac{5}{1-m})$, and $(0, 5)$. Writing these terms in shoelace format:

$$\begin{array}{c|c} 0 & 0 \\ \frac{5}{1-m} & \frac{5}{1-m} \\ 0 & 5 \\ 0 & 0 \end{array}$$

Cross-multiplying from top left to bottom right yields $\frac{25}{1-m}$. Cross-multiplying from top right to bottom left yields 0. Averaging the two yields $\frac{25}{2-2m}$. This is the area of the triangle given, in terms of m.

To find the value of m such that the area equals 10, we can set up an equation:

$$\frac{25}{2-2m} = 10$$
$$\frac{5}{2} = 2 - 2m$$
$$2m = -\frac{1}{2}$$
$$m = -\frac{1}{4}$$

Thus, $\boxed{m = -\frac{1}{4}}$.

4.5 Context: Problem 4.5. The (average) sale price for single family property in Seattle and Port Townsend is tabulated below:

Year	**Seattle**	**Port Townsend**
1970	\$38,000	\$8400
1990	\$175,000	\$168,400

4.5a Problem: Find a linear model relating the year x and the sales price y for a single family property in Seattle.

4.5a Solution: Two points given are $(1970, 38000)$ and $(1990, 175000)$ for average sale price for single family property in Seattle. The slope is $\frac{175000-38000}{1990-1970} = \frac{137000}{20} = 6850$. Writing in point-slope form:

$$y - 38000 = 6850(x - 1970)$$
$$y = 6850(x - 1970) + 38000$$

Thus, the equation is $\boxed{y = 6850(x - 1970) + 38000.}$

4.5b Problem: Find a linear model relating the year x and the sales price y for a single family property in Port Townsend.

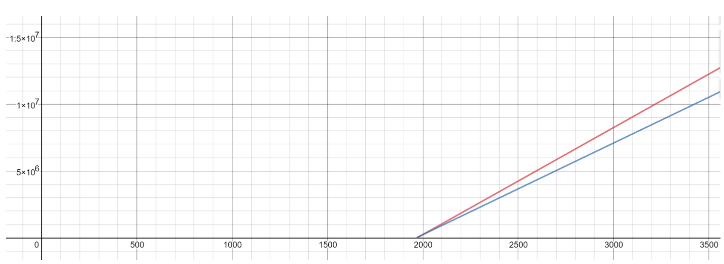
4.5b Solution: The two points given are (1970, 8400) and (1990, 168400). The slope is $\frac{168400-8400}{1990-1970} = \frac{160000}{20} = 8000$. Writing in point-slope form:

$$y - 8400 = 8000(x - 1970)$$
$$y = 8000(x - 1970) + 8400$$

Thus, the equation is $\boxed{y = 8000(x - 1970) + 8400}$.

4.5c Problem: Sketch the graph of both modeling equations in a common coordinate system; restrict your attention to $x \geq 1970$.

4.5c Solution: Red line for Port Townsend, blue line for Seattle.



4.5d Problem: What is the sales price in Seattle and Port Townsend in 1983 and 1998?

4.5d Solution: This is simply a series of calculations, using our derived models. Let $p_s(x)$ be the price of a house in Seattle given year x, and let $p_t(x)$ be the price of a house in Port Townsend given year x.

$$\begin{aligned}
\text{Price in 1983 of Seattle} \rightarrow p_s(1983) &= 6850(1983 - 1970) + 38000 = \$127050 \\
\text{Price in 1983 of Port Townsend} \rightarrow p_t(1983) &= 8000(1983 - 1970) + 8400 = \$112400 \\
\text{Price in 1998 of Seattle} \rightarrow p_s(1998) &= 6850(1998 - 1970) + 38000 = \$229800 \\
\text{Price in 1998 of Port Townsend} \rightarrow p_t(1998) &= 8000(1998 - 1970) + 8400 = \$232400
\end{aligned}$$

Tabulating answers:

Year	Price in Seattle	Price in Port Townsend
1983	\$127,050	\$112,400
1998	\$229,800	\$232,400

4.5e Problem: When will the average sales price in Seattle and Port Townsend be equal and what is this price?

4.5e Solution: To solve this problem, we need to find a year x such that $p_s(x) = p_t(x)$.

$$\begin{aligned}
6,850(x - 1,970) + 38,000 &= 8,000(x - 1,970) + 8,400 \\
6850x - 13,456,500 &= 8000x - 15,751,600 \\
68.5x - 134,565 &= 80x - 157,516 \\
11.5x &= 22,951 \\
x &\approx 1995.7391
\end{aligned}$$

Finding the value at year 1995.7391:

$$\begin{aligned}
p_s(1995.7391) = p_t(1995.7391) &= 8,000(1995.7391 - 1,970) + 8,400 \\
&= 214312.8
\end{aligned}$$

Thus, at $\boxed{\text{year 1995.7391 and price \$214,313}}$, the prices for Port Townsend and Seattle are the same.

4.5f Problem: When will the average sales price in Port Townsend be \$15,000 less than the Seattle sales price? What are the two sales prices at this time?

4.5f Solution: This question asks for a year x such that $p_t(x) + 15,000 = p_s(x)$.

$$\begin{aligned}
8,000(x - 1,970) + 8,400 + 15,000 &= 6,850(x - 1,970) + 38,000 \\
8,000x - 15,736,600 &= 6850x - 13,456,500 \\
1,150x &= 2,280,100 \\
x &\approx 1982.6957
\end{aligned}$$

Finding the two values:

$$\begin{aligned}
p_s(1982.6957) &= 6,850(1982.6957 - 1,970) + 38,000 \\
&= 124,965.545
\end{aligned}$$

$$\begin{aligned}
p_t(1982.6957) &= 8,000(1982.6957 - 1,970) + 8,400) \\
&= 109,965.6
\end{aligned}$$

Thus, at $\boxed{\text{year 1982.7, Seattle price at \$124,965 and Port Townsend price at \$109,966}}$ the Seattle price is $15,000 larger than that of Port Townsend's.

4.5g Problem: When will the Port Townsend sales price be $15,000 more than the Seattle sales price? What are the two sales prices at this time?

4.5g Solution: This question asks for a year x such that $p_t(x) = p_s(x) + 15,000$.

$$\begin{aligned} 8,000(x - 1,970) + 8,400 &= 6,850(x - 1,970) + 38,000 + 15,000 \\ 8,000x - 15,751,600 &= 6,850x - 13,441,500 \\ 1,150x &= 2,310,100 \\ x &\approx 2008.7826 \end{aligned}$$

Finding the two values:

$$\begin{aligned} p_s(1982.6957) &= 6,850(2008.7826 - 1,970) + 38,000 \\ &= 303,660.81 \end{aligned}$$

$$\begin{aligned} p_t(1982.6957) &= 8,000(2008.7826 - 1,970) + 8,400) \\ &= 318,660.8 \end{aligned}$$

Thus, at $\boxed{\text{year 2008.78, Seattle price at \$303,660.81 and Port Townsend price at \$318,660.8}}$ the Seattle price is $15,000 larger than that of Port Townsend's.

4.5h Problem: When will the Seattle sales price be double the Port Townsend sales price?

4.5h Solution: This question asks for a year x such that $2 \cdot p_t(x) = p_s(x)$.

$$\begin{aligned} 2(8,000(x - 1,970) + 8,400) &= 6,850(x - 1,970) + 38,000 \\ 16,000x - 31,503,200 &= 6,850x - 13,456,500 \\ 9150x &= 18,046,700 \\ x &\approx 1972.3169 \end{aligned}$$

Finding the two values:

$$\begin{aligned} p_s(1972.3169) &= 6,850(1972.3169 - 1,970) + 38,000 \\ &= 53,870.765 \end{aligned}$$

$$\begin{aligned} p_t(1972.3169) &= 8,000(1972.3169 - 1,970) + 8,400) \\ &= 26,935.2 \end{aligned}$$

Thus, at $\boxed{\text{year 1972.32, Seattle price at \$53,870.765 and Port Townsend price at \$26,935.2}}$ the Seattle price is $15,000 larger than that of Port Townsend's. *These numbers are a little bit off because the textbook plugs in the approximation for the date, 1972.32, instead of a more accurate four-decimal space representation.*

4.5i Problem: Is the Port Townsend sales price ever double the Seattle sales price?

4.5h Solution: This question asks for a year x such that $p_t(x) = 2 \cdot p_s(x)$.

$$\begin{aligned} 8,000(x - 1,970) + 8,400 &= 2(6,850(x - 1,970) + 38,000) \\ 8,000x - 15,751,600 &= 13,700x - 26,913,000 \\ 11,161,400 &= 5700x \\ x &\approx 1958.1404 \end{aligned}$$

Plugging this derived year into one of the equations for verification, we find that $p_t(1958.1404) = 8,000(1958.1404 - 1,970) + 8,400 = -86476.8$. Since the price of a house cannot be negative (unless you have very undesirable land and very generous sellers), $\boxed{\text{the Port Townsend price is never double the Seattle price.}}$