Collingwood 11

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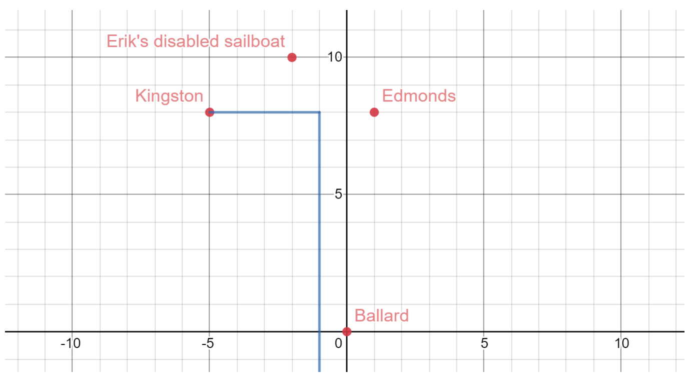
3.6 Context: Erik's disabled sailboat is floating stationary 3 miles East and 2 miles North of Kingston. A ferry leaves Kingston heading toward Edmonds at 12 mph. Edmonds is 6 miles due east of Kingston. After 20 minutes the ferry turns heading due South. Ballard is 8 miles South and 1 mile West of Edmonds. Impose coordinates with Ballard as the origin.

3.6a Problem: Find the equations for the lines along which the ferry is moving and draw in these lines.

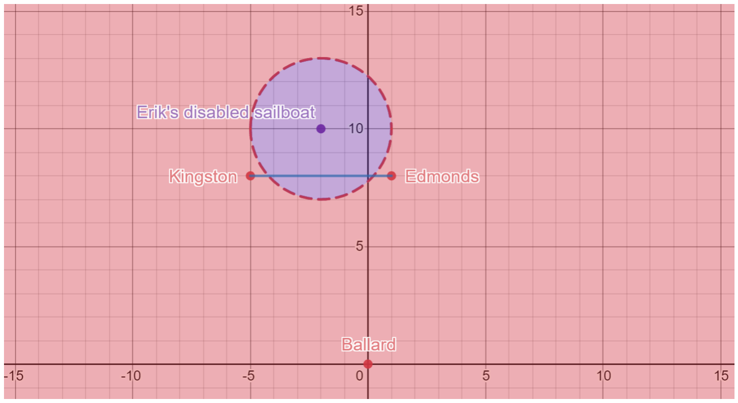
3.6a Solution: If Ballard is located at the origin (0,0), Edmonds is located at (1, 8), and Kingston is located at (-5, 8). Erik's disabled sailboat is floating at (-2, 10). The west-east travel follows $\boxed{y = 8}$. Since the ferry heads east at 12 mph, at 20 minutes ($\frac{1}{3}$ hour) the ferry has travelled 4 miles east from Kingston at location (-1, 8). Thus, the north-south travel follows $\boxed{x = -1}$.



3.6b Problem: The sailboat has a radar scope that will detect any object within 3 miles of the sailboat. Looking down from above, as in the picture, the radar region looks like a circular disk. The boundary is the "edge" or circle around this disc, the interior is the inside of the disk, and the exterior is everything outside of the disk (i.e. outside of the circle). Give a mathematical (equation) description of the boundary, interior and exterior of the radar zone. Sketch an accurate picture of the radar zone by determining where the line connecting Kingston and Edmonds would cross the radar zone.

3.6b Solution: In our imposed coordinate system, the sailboat is located at point (-2, 10). Thus, the equation for a circle of radius 3 miles would be $\boxed{(x + 2)^2 + (y - 10)^2 = 9}$. Correspondingly, we can represent the interior and exterior with inequalities:

- Interior: $\boxed{(x + 2)^2 + (y - 10)^2 < 9}$
- Exterior: $\boxed{(x + 2)^2 + (y - 10)^2 > 9}$



3.6c Problem: When does the ferry enter the radar zone?

3.6c Solution: The ferry enters the radar zone along the east-west path $y = 8$. We can use this value of y in the circle equation for the radar.

$$\begin{aligned}(x+2)^2 + (y-10)^2 &= 9\\(x+2)^2 + (-2)^2 &= 9\\(x+2)^2 &= 5\\x+2 &= \pm\sqrt{5}\\x &= \pm\sqrt{5} - 2\end{aligned}$$

Since the ferry enters the radar on the left side, we choose the smaller value of x, which is $-\sqrt{5} - 2 \approx -4.24$ on the coordinate system. Since the ferry begins at location -5, it travels ≈ 0.76 miles before entering the radar. Since the ferry runs at 12 miles per hour, it takes $\frac{0.76}{12} \approx 0.063$ hours, or $0.063 \cdot 60 = 3.8$ minutes. Therefore, the ferry takes $\boxed{\text{3.8 minutes}}$ to enter the radar.

3.6d Problem: Where and when does the ferry exit the radar zone?

3.6d Solution: Since the ferry exits the ferry zone along the north-south path $x = -1$, we can find the y-values for which $x = -1$ in the equation for the radar circle.

$$\begin{aligned}(x+2)^2 + (y-10)^2 &= 9\\(-1+2)^2 + (y-10)^2 &= 9\\1 + (y-10)^2 &= 9\\(y-10)^2 &= 8\\y - 10 &= \pm 2\sqrt{2}\\y &= \pm 2\sqrt{2} + 10\end{aligned}$$

Since from the diagram we see that we are looking for the southernmost value of y, we choose the smaller value, $-2\sqrt{2} + 10 \approx 7.17$. Therefore, the ferry exits the radar zone at location $\boxed{\text{(-1, 7.17)}}$.

To find how long the ferry took to exit the radar, we must calculate the distance it travelled. The ferry travels 4 miles from (-5,8) - Kingston - to the location (-1,8). Afterwards, it travels south 0.8284 miles to (-1, 7.17), exiting the radar. In total, the ferry travels ≈ 4.8284 miles. Thus, travelling at a speed of 12 miles per hour, it takes $\frac{4.8284}{12} \approx 0.4024$ hours, or $0.4024 \cdot 60 = 24.1421$ minutes. Hence, it takes about $\boxed{\text{24.14 minutes}}$ for the ferry to exit the radar.

3.6e Problem: How long does the ferry spend inside the radar zone?

3.6e Solution: To find this, we simply subtract the time the ferry took to enter the radar form the time the ferry took to exit the radar. This comes out to be $24.1421 - 3.8 = 20.3421 \approx 20.3$. Thus, the ferry spends about $\boxed{\text{20.3 minutes}}$ inside the radar zone.

3.7 Context: Nora spends part of her summer driving a combine during the wheat harvest. Assume she starts at the indicated position heading east at 10 ft/sec toward a circular wheat field of radius 200 ft. The combine cuts a swath 20 feet wide and begins when the corner of the machine labeled "a" is 60 feet north and 60 feet west of the western-most edge of the field.

3.7a Problem: When does Nora's rig first start cutting the wheat?

3.7a Solution: Let the westernmost edge of the field be the point (0,0). Thus, Nora's combine (the south-east corner) begins at location (-60, 60) and travels along the horizontal line represented by $y = 60$. The equation of a circle representing the circular wheat field is then $(x - 200)^2 + y^2 = 40,000$. Setting the $y = 60$, we obtain values for x at which Nora's combine touches the edges of the wheat field.

$$\begin{aligned}
(x-200)^2 + y^2 &= 40,000 \\
(x-200)^2 + 3,600 &= 40,000 \\
(x-200)^2 &= 36,400 \\
x - 200 &= \pm\sqrt{36,400} \\
x &= \pm\sqrt{36,400} + 200
\end{aligned}$$

Because Nora's combine enters the circular wheat field from the west side, we want the smaller value of x, which is $-\sqrt{36,400} + 200 \approx 9.2122$. Starting from $x = -60$, Nora's combine travels 69.2122 feet. Since Nora travels at 10 ft/sec on her combine, she can travel that distance in $\frac{69.2122}{10} \approx 6.9212$ minutes. Thus, Nora's combine first begins cutting the wheat after $\boxed{\text{6.92 seconds.}}$

3.7b Problem: When does Nora's rig first start cutting a swath 20 feet wide?

3.7b Solution: To find when Nora's rig first begins cutting a swath 20 feet wide, we need to track when the north-east corner of Nora's rig touches the field. This is because it is the last part of the front of rig to enter the field, and the entire front of the rig must be in the field for it to begin cutting a swath 20 feet wide, which is its length. Since Nora's south-east corner is located at (-60, 60), the north-east corner is located at (-60, 80). Thus, it follows the path $y = 80$. Plugging this into the circle equation that represents the border of the circular wheat field gives us values of x for which we can calculate time.

$$\begin{aligned}
(x-200)^2 + y^2 &= 40,000 \\
(x-200)^2 + 6,400 &= 40,000 \\
(x-200)^2 &= 33,600 \\
x - 200 &= \pm\sqrt{33,600} \\
x &= \pm\sqrt{33,600} + 200
\end{aligned}$$

Since the corner enters the wheat field from the west side, we want a negative value of x. This comes out to be $-\sqrt{33,600} + 200 \approx 16.697$. Since the corner begins at $x = -60$, it travels 76.697 feet. Nora's rig travels at 10 feet per second, so it can travel that distance in $\frac{76.697}{10} \approx 7.6697$ seconds. Thus, Nora's rig begins cutting a swath twenty feet wide after $\boxed{\text{7.67 seconds}}$.

3.7c Problem: Find the total amount of time wheat is being cut during this pass across the field.

3.7c Solution: To track the total amount of time wheat is being cut while it passes through the field, we need to find the moment when the combine first enters the field and the time when the combine completely exits the field. We already have obtained the former from problem 3.7a; the combine first begins cutting wheat at 6.92 minutes.

Wheat stops being cut when the south-east corner of the combine exits the field. In problem 3.7a, we obtained two solutions. Because we wanted to find when the combine entered the field, we chose the smaller one. In this case, we want to choose the large one to find when the combine exits the field; this comes to be $\sqrt{36,400}+200 \approx 390.7878$. Since the south-east corner begins at $x = -60$, it travels 450.7878 feet. Travelling at a rate of 10 feet per seconds, Nora's rig completes the distance in $\frac{450.7878}{10} \approx 45.0788$ seconds.

Taking the difference between the two times yields 38.1588 seconds. Hence, it takes $\boxed{\text{38.16 seconds}}$ for the combine to cut through all of the wheat.

3.7d Problem: Estimate the area of the swath cut during this pass across the field.

3.7d Solution: The area of the swath cut during the pass is somewhere between a rectangle whose length is determined by the starting and ending points of the northeast corner of the combine and a rectangle whose length is determined by the starting and ending points of the southeast corner of the combine.

The starting and ending points of the northeast corner are $-\sqrt{36,400} + 200$ and $\sqrt{36,400} + 200$. The distance between these two points is $2 \cdot \sqrt{36,400} = 381.5757$ feet. The starting and ending points of the southeast corner are $-\sqrt{33,600} + 200$ and $\sqrt{33,600} + 200$. The distance between these two points is $2 \cdot \sqrt{33,600} = 366.6061$ feet. Because the swath is 20 feet wide, the area of the rectangles are 7631.5140 and 7332.12 square feet, respectively.

Thus, the area of the swath cut is $\boxed{\text{between 7332 and 7632 square feet}}$.