# The Continued Fraction Problem

Ayush Agrawal, Vivek Prakriya, Austin Shaw, Andre Ye

## Contents

# 1 Introduction

Continued fractions are a method of describing non-integer numbers in a far more natural and elegant way than, say, the arbitrary base-10 decimal expansion conventionally used. This is because while base-10 (or any base) uses an arbitrary value of ten or the like, a continued fraction's expression is functionally identical over any base. For example, let us consider the example of $\frac{355}{113}$. The decimal expansion of this fraction is

$$3.14159292035398230088495575221238 93...$$

and repeats with period 112. Now, the continued fraction representation,

$$3 + \cfrac{1}{7 + \frac{1}{16}},$$

looks far more elegant and simple than the rather unruly decimal expansion. Irrational numbers may expressed as *infinite* continued fractions, of the form,

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}.$$

Our problem will investigate infinite continued fractions where $a_0 = a_1 = a_2 \dots$, or to put it more concisely,

$$a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{\ddots}}}.$$

Furthermore, we can represent this as an infinite compounding of functions. Let $f$ be a function such that $f(x) = a + \frac{1}{x}$, thus our infinite continued fraction can be expressed as $f(f(f(f(\cdots))))$.

We will explore a series of questions, finding that each leads us to new ones, to better understand the nature of this infinite fraction. As such, these questions may be confusing initially, but these will be further elaborated upon.

1. What values do this infinite fraction take on? If we express it by iterating $f$ many times, what values does that repeated iteration converge to?
2. What are the value of the stable and unstable fixed points the infinite fraction converges to, given $a$?
3. Prove that the stable fixed point of $f(x)$ is the unstable fixed point of $f^{-1}(x)$, and vice versa.

# 2 Exploration of infinite continued fractions as infinite compositions

We can start by exploring this problem, plugging in a few simple values of $a$ and then looking at our findings. We start with perhaps the simplest value, $a = 0$. When we have that $a = 0$, our fractional representation is

$$0 + \cfrac{1}{0 + \cfrac{1}{0 + \cfrac{1}{\ddots}}}$$

We can see that at any stoppage point, this is the number 1 divided by itself numerous times, so the expression is equal to 1. We will explore the complexity of this later, because $a = 0$ leads to some problematic and interesting results.

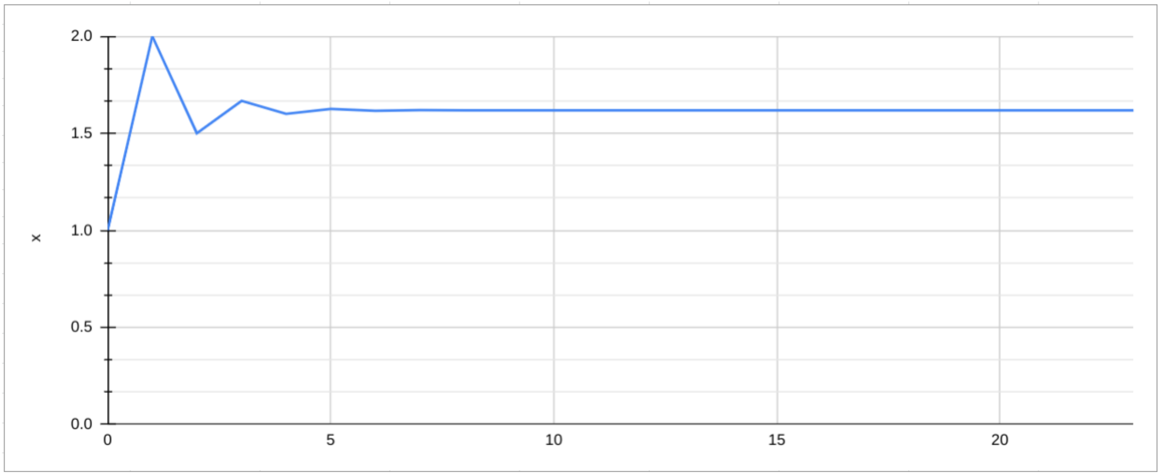


Figure 2.1: When $a = 1$, repeated composition of the function converges to $\phi$.

A very interesting result happens when we plug in $a = 1$; the graph of the first 25 iterations is presented in Figure 1.

After these 25 iterations, it converges to around 1.618; interestingly, this value reminds one of the golden ratio, $\phi = \frac{1+\sqrt{5}}{2}$. When we look at the first few iterations, we see the values of $\frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}$. We see that each iteration is the ratio of the next two consecutive numbers in the Fibonacci sequence. As $n$ approaches infinity, $\frac{f_{n+1}}{f_n} = \phi$, so this confirms our suspicions about the golden ratio.

We try another simple value, $a = 2$. This gives the composition:

$$2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}$$

After 25 iterations at $a = 2$, the value of the composition comes out to 2.4142; we recognize that this is nearly equivalent to the $\sqrt{2} + 1$, and indeed it is. So far, both of our initial trials have converged to values revolving around radicals.

To get some more insight, we try to look at where the continued fraction converges to for values of $a$ that are very large and very small. We can see that for larger and larger values of $a$, it converges to $a$ more and more closely. For example, when $a = 10$ it converges to 10.09, and when $a = 100$ it converges to 100.01, very close to the initial value of $a$. This makes sense, considering the fact that for large values of $a$, the fractional part of

$$a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{\ddots}}}$$

gets smaller and smaller, leaving only the $a$ at the beginning to have significance. The same logic applies to negative values of $a$ that have a very large absolute value. We now look to values of $a$ that are very small. The graphs of such $a$ look somewhat like figure 2.

This is the graph when $a = 0.01$. As we can see, it converges to around the value of 1, which is what happens when $a$ gets closer and closer to 0, either positive or negative. Informally, we can explain this by seeing that the initial $a$ value in

$$a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{\ddots}}}$$

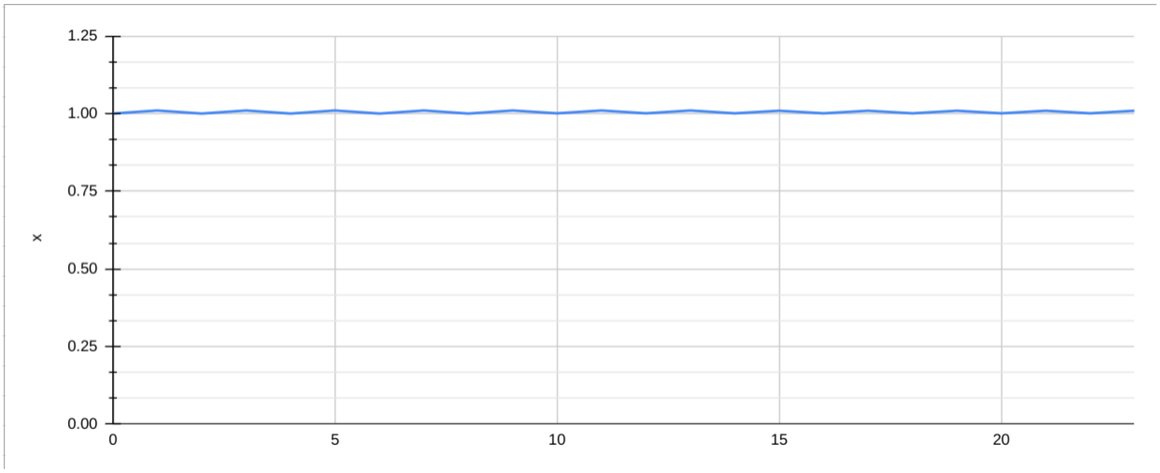


Figure 2.2: When $a = 0.01$, repeated composition of the function converges to 1.

It is negligible as $a$ goes to 0, and the denominator of the fractional part goes to around 1, yielding the convergence of 1.

The following sections will explain these interesting observations.

# 3 Solving for convergence values

## 3.1 Solving for and explaining $f(x)$ convergence

Let

$$x = a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{\ddots}}},$$

where $x$ is the convergence value. This gives

$$x = a + \frac{1}{x} \implies x^2 - ax - 1 = 0 \implies x = \frac{a \pm \sqrt{a^2 + 4}}{2}.$$

An interesting (but unrelated to broader conversation) observation is that the sum of the two possible convergence values is $a$. The formula for the convergence values was found to be $\frac{a \pm \sqrt{a^2+4}}{2}$. Adding these two solutions yields

$$\frac{a + \sqrt{a^2 + 4}}{2} + \frac{a - \sqrt{a^2 + 4}}{2} = \frac{2a}{2} = a$$

What is puzzling about this result is that there are in fact *two* solutions to convergence (given by the $\pm$ in the formula for convergence value), even though through simulations and intuition it would seem that there would only be one.

In the computer simulation, there is a starting value, $s$; the function $f(x) = a + \frac{1}{a}$ is iterated upon it such that $f^\infty(s)$ serves as being approximately equivalent to the infinite fraction $a + \frac{1}{a + \frac{1}{a + \dots}}$. (Note in the context of computed simulation, '$\infty$' really means 'a large number such that convergence appears to have been reached'.) Thus, the value of $s$ - the initial starting value - is generally irrelevant as it becomes 'lost in the cascade of infinite fractions', which converges to a value dependent on $a$. However, in our simulation we found two values of $s$ for which the iteration does *not* converge to the 'general' convergence solution $c_g$: $s = -a$ and $s = c_s$ (the 'singular' convergence solution).

In the first instance, it is obvious that an initial value $s$ that is equal to $-a$ causes division-by-zero errors. On the second instance, $c_g$ and $c_s$ are both solutions to the previously derived $\frac{a\pm\sqrt{a^2+4}}{2}$. However, $c_s$ has a property such that $f(c_s) = c_s$, and repeated iteration of the function does not change its value - this does make it a 'fixed point' and a mathematically valid solution for convergence. On the other hand, $c_g$ has the property such that $f^{\infty}(k) = c_g\{k \in \mathbb{R} | k \notin c_s, -a\}$. More formal language and constructions around 'general' and 'singular' solutions will be discussed later.

Importantly, we observe from simulations that the inverse function, $f^{-1}$ is such that it shares the same solutions for convergence as $f$, but that *$f$'s general solutions are the singular solutions of $f^{-1}$, and $f^{-1}$'s singular solutions are the general solutions of $f$.*

### 3.2 Solving for $f^{-1}(x)$ convergence

Solving for the inverse of $f(x) = a + \frac{1}{x}$ gives that $f^{-1}(x) = \frac{1}{x-a}$. Compounding $f^{-1}(x)$ infinitely, and setting this equal to a convergence value $x$ gives

$$x = \cfrac{1}{-a + \cfrac{1}{-a + \cfrac{1}{\ddots}}}.$$

This leads to

$$x = \frac{1}{x-a} \implies x^2 - ax - 1 = 0 \implies x = \frac{a \pm \sqrt{a^2+4}}{2}.$$

This is the same formula as for convergence values of $f$. This leads one to hypothesize that, given the same value of $a$, $f$ and $f^{-1}$ take different signs from the $\pm$ in the formula in their $c_g$ and $c_s$ solutions. The next sections serve to better understand the dynamics of these two solutions, and to prove this hypothesis.

## 4 Convergence and stability

One can observe that while there are two solutions for convergence, the graph only converges towards one value. That is, out of the two fixed point solutions, one is 'stable' and the other is 'unstable'; this replaces earlier language about 'general' and 'singular' solutions, respectively.

### 4.1 Theory of stable points

For a function defined iteratively: $x_{n+1} = f(x_n)$ (such as our iterated continued fraction), there may exist points such that $f(x) = x$, these are, again, the fixed points. We can further classify fixed points into *stable* and *unstable* fixed points: a *stable* fixed point is where points "near" the fixed point move toward the point as the function is iterated. In mathematical terms, $x$ is a fixed point if $|f(x+\Delta x) - f(x)| < \Delta x$, where $\Delta x$ is a nonzero but sufficiently small quantity. Therefore, multiple iterations will bring two points near a fixed point arbitrarily close together. An unstable point is the opposite: a point where $|f(x+\Delta x) - f(x)| > \Delta x$, so multiple iterations will push points farther apart.

Reiterating the definition of a stable fixed point, $x$ is stable if $|f(x+\Delta x) - f(x)| < \Delta x$, or

$$\frac{|f(x+\Delta x) - f(x)|}{\Delta x} < 1$$

(with the less than becoming a greater than for unstable points). As $\Delta x$ approaches zero, we obtain

$$\frac{|f(x+dx) - f(x)|}{dx} < 1 \implies \left|\frac{df}{dx}\right| < 1.$$

This means that if the absolute value of the derivative of $f$ at a fixed point $x$ is less than one, then $x$ is a stable fixed point.

## 4.2 Obtaining conditions for stable fixed point convergence

The derivative of the function $f(x) = a + \frac{1}{x}$ can be found as follows:

$$\begin{aligned} \frac{d}{dx}\left(a + \frac{1}{x}\right) &= \frac{d}{dx}(a) + \frac{d}{dx}\left(\frac{1}{x}\right) \\ &= 0 - \frac{1}{x^2} \\ &= -\frac{1}{x^2} \end{aligned}$$

As established, if $|f'(b)| < 1$ for some fixed point $b$ of $f(x)$, it is stable. Solving for when the derivative is less than 1:

$$\begin{aligned} \left|-\frac{1}{b^2}\right| &< 1 \\ 1 &< b^2 \\ |b| &> 1 \end{aligned}$$

Thus, for fixed point solutions whose absolute value is larger than 1, the point is stable. Simulations confirm this result; for instance, in the figure below, $a = 0.4$. The two solutions for convergence, according to the formula derived in section 3.1, are $\approx 1.2198$ and $\approx -0.8198$. Iterating $f$ converges to the former due to the derivative at that point being less than one, as shown in figure 3.

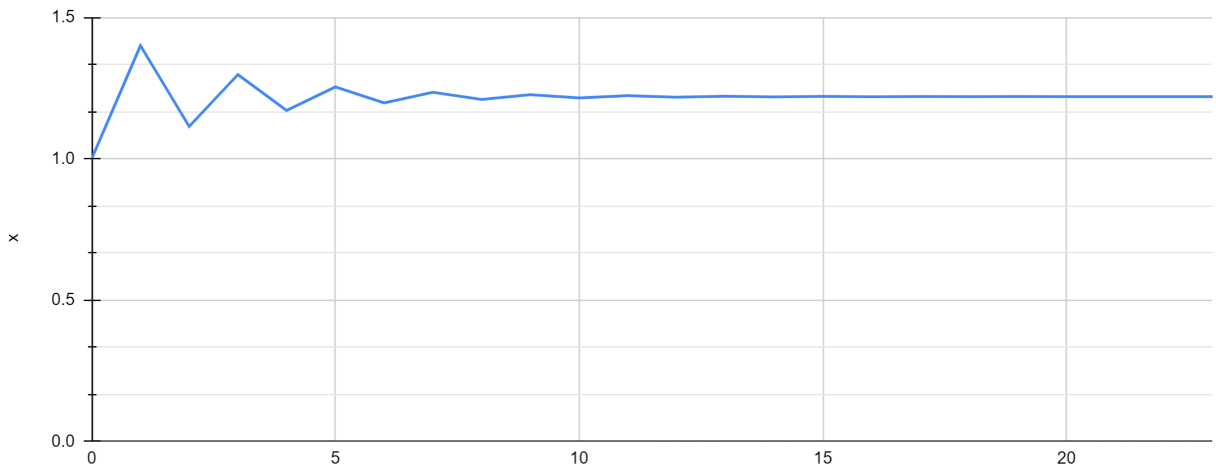


Figure 4.1: Iteration of $f$ with $a = 0.4$.

The derivative of the *inverse function* $f^{-1}(x) = \left(\frac{1}{x-a}\right)$, which was established to have stable fixed points where $f(x)$ has unstable fixed points, can be similarly found:

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{x-a}\right) &= \frac{d}{dx}\left((x-a)^{-1}\right) \\ &= -\frac{1}{(x-a)^2}\frac{d}{dx}(x-a) \\ &= -\frac{1}{(x-a)^2} \end{aligned}$$

Solving for when the absolute value of the derivative is less than 1:

$$\left| -\frac{1}{(b-a)^2} \right| < 1$$
$$\frac{1}{(b-a)^2} < 1$$
$$1 < (b-a)^2$$
$$1 < |b-a|$$

Thus, for fixed point solutions $b$, such that the absolute value of $b-a$ is larger than 1, the point is stable. Simulations confirm this result; for instance, in the figure below, $a = 0.4$. The two solutions for convergence, according to the formula derived in section 3.2, are $\approx 1.2198$ and $\approx -0.8198$. Using the $1 < |b-a|$ condition, we find:

- For solution $b = 1.2198$, $|b-a| = |1.2198 - 0.4| = 0.8198$. This does not satisfy the condition $1 < 0.8198$.
- For solution $b = -0.8198$, $|b-a| = |-0.8198 - 0.4| = 1.2198$. This satisfies the condition $1.2198 > 1$.

Iterating $f^{-1}$ confirms that the result converges to the second solution, $-0.8198$.

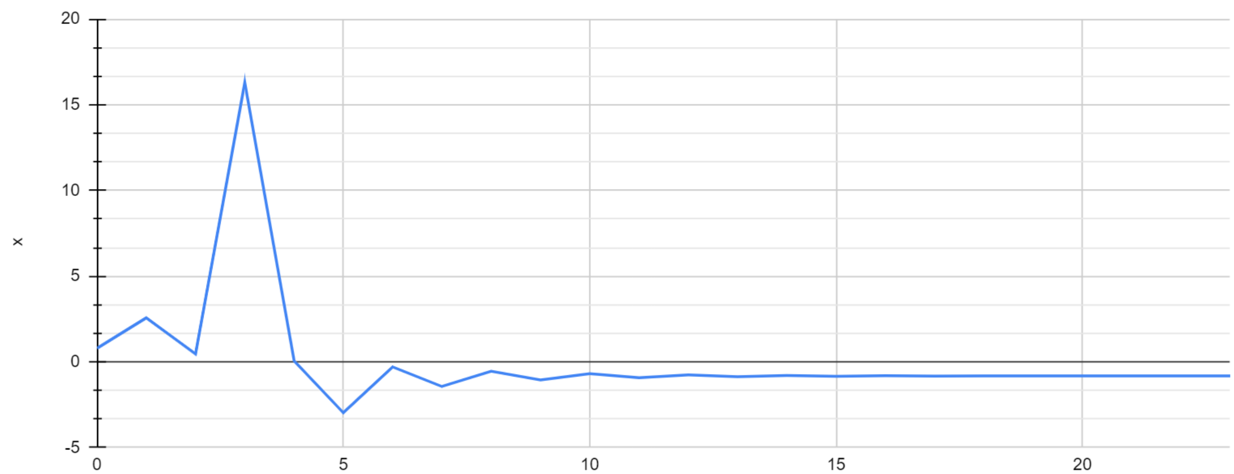


Figure 4.2: Iteration of $f^{-1}$ with $a = 0.4$.

# 5 Deriving ±-absent formulae for stable fixed point convergence value

We have found that the formulas for convergence value, $x = \frac{a \pm \sqrt{a^2+4}}{2}$, gives *two* solutions. Furthermore, we have derived conditions for which a fixed point is stable or unstable. To further advance these findings, in this section we will find under which conditions to take the '+' or the '−' from this formula such that the value is the stable fixed point of infinite composition of $f$.

## 5.1 For $f(x)$

Recall that the solutions for the convergence value of the infinite fraction is $x = \frac{a \pm \sqrt{a^2+4}}{2}$. We derived in the previous sections that for this to be the *stable fixed point* of the infinite repeating fraction, it must satisfy $|x| > 1$. Using substitution, this means that the following must be true:

$$\left|\frac{a \pm \sqrt{a^2+4}}{2}\right| > 1$$

Our goal is to understand under which conditions of $a$ should the $+$ or $-$ of the $\pm$ in the formula be used such that it converges to the stable fixed point. We can do this by establishing the truth of various mathematical statements when $a > 0$, $a < 0$, and $a = 0$ and the $+$ or $-$ is taken.

### 5.1.1 When $a > 0$

When $a > 0$, there are two solutions for convergence: $\frac{a+\sqrt{a^2+4}}{2}$ and $\frac{a-\sqrt{a^2+4}}{2}$. This results in two mathematical statements:

$$\left|\frac{a + \sqrt{a^2+4}}{2}\right| > 1 \text{ and } \left|\frac{a - \sqrt{a^2+4}}{2}\right| > 1$$

The first statement, taking the $+$ from $\pm$, is true. We can prove it as such:

$$\left|\frac{a + \sqrt{a^2+4}}{2}\right| > 1$$

$$\left|a + \sqrt{a^2+4}\right| > 2$$

$$a + \sqrt{a^2+4} > 2$$

We can remove the absolute value around the left-hand side because $a > 0$ and $\sqrt{a^2+4}$ is positive; thus the absolute value is redundant. Furthermore, we see that at $a = 0$, the expression $a + \sqrt{a^2+4}$ is *equal* to two; since the two terms of the additive expression, $a$ and $\sqrt{a^2+4}$, increase monotonically, we can assume that for $a > 0$, $a + \sqrt{a^2+4} > 2$.

The second statement, taking the $-$ from $\pm$, is not true. We can prove it as such:

$$\left|\frac{a - \sqrt{a^2+4}}{2}\right| > 1$$

$$\left|a - \sqrt{a^2+4}\right| > 2$$

Absolute value allows us to change the order of the two terms in the LHS:

$$\left|\sqrt{a^2+4} - a\right| > 2$$

For values $a \geq 0$, one can reason that $\sqrt{a^2+4}$ will always be larger than $a$. By squaring a number, adding some number to it, and square-rooting it, that number will become larger. Furthermore, one can reason that as $a$ gets larger, the difference between $\sqrt{a^2+4}$ and $a$ grows smaller; the significance of adding a constant, 4, becomes less important relative to the size of $a$. Thus, it can be concluded that $\sqrt{a^2+4} - a$ grows smaller as $a$ increases.

For a more formal proof of this assertion, it is true that

$$\frac{d}{da}\left(\sqrt{a^2+4} - a\right) = \frac{d}{da}\left(\sqrt{a^2+4}\right) - \frac{d}{da}(x) = \frac{a}{\sqrt{a^2+4}} - 1.$$

This derivative is negative for any value of $a$, since $\frac{a}{\sqrt{a^2+4}}$ can never be larger than 1. Hence, $\sqrt{a^2+4} - a$ monotonically decreases as $a$ is larger.

At $a = 0$, the expression evaluates to 2. Given that the expression decreases for any larger value of $a$ (such that $a > 0$), the expression will never be larger than 2 given the domain restriction.

Thus, when $a > 0$, taking the '+' from $\pm$ gives the stable fixed point.

### 5.1.2 When $a < 0$

We need to see the truth of the inequalities

$$\left|\frac{a+\sqrt{a^2+4}}{2}\right| > 1 \text{ and } \left|\frac{a-\sqrt{a^2+4}}{2}\right| > 1$$

when $a < 0$. To make things simpler, we establish a variable $b$ such that $b = -a$. Since $a < 0$, we know that $b > 0$. We start off with the $-$ here. We therefore have to evaluate the truth of

$$\left|\frac{-b-\sqrt{b^2+4}}{2}\right| > 1 \implies \left|-b-\sqrt{b^2+4}\right| > 2$$

Immediately, we recognize that $-b-\sqrt{b^2+4}$ must be negative, because both $-b$ and $-\sqrt{b^2+4}$ are both negative. Therefore, for this inequality to be true, we must have that $-b-\sqrt{b^2+4} < -2$ for the absolute value. We know that $-b < 0$. Additionally, we know that $\sqrt{b^2+4} > 2$ when $b > 0$ from reasoning provided in the last subsection, so $-\sqrt{b^2+4} < -2$. Therefore, we have that $-b-\sqrt{b^2+4} < -2$ is true and thus that the inequality $\left|\frac{a-\sqrt{a^2+4}}{2}\right| > 1$ is true.

Taking the $+$, we are presented that

$$\left|\frac{a+\sqrt{a^2+4}}{2}\right| > 1 \implies \left|\frac{-b+\sqrt{b^2+4}}{2}\right| > 1 \implies \left|\sqrt{b^2+4}-b\right| > 2$$

The previous subsection establishes that for $a > 0$, the inequality

$$\left|\sqrt{a^2+4}-a\right| > 2$$

is false. Therefore, when $b > 0$

$$\left|\sqrt{b^2+4}-b\right| > 2$$

is also false, which means that taking the $+$ does not work. Therefore, when $a < 0$, only the $-$ from the $\pm$ yields the stable fixed point.

## 5.2 For $f^{-1}(x)$

We determined that the solutions for the convergence value of the infinite fraction is $x = \frac{a \pm \sqrt{a^2+4}}{2}$. In order for a value of $x$ to be a stable fixed point, we found that it must satisfy $|x-a| > 1$. Substituting gives us

$$\left|\frac{a \pm \sqrt{a^2+4}}{2} - a\right| > 1.$$

We can simplify the left side of the inequality to give us

$$\frac{\left|-a \pm \sqrt{a^2+4}\right|}{|2|} > 1 \implies \left|-a \pm \sqrt{a^2+4}\right| > 2.$$

Our goal is to understand under which conditions of $a$ should the $+$ or $-$ of the $\pm$ in the formula be used such that it converges to the stable fixed point. We can do this by establishing the truth of various mathematical statements when $a > 0$, $a < 0$, and $a = 0$ and the $+$ or $-$ is taken.

#### 5.2.1 When $a > 0$

For $a > 0$, setting the $\pm$ as $+$ yields the inequality

$$\left|-a + \sqrt{a^2+4}\right| > 2,$$

which we have previously established to be false for positive $a$.

However, setting the $\pm$ as $-$ yields the inequality

$$\left|-a - \sqrt{a^2+4}\right| > 2.$$

Immediately, we recognize that $-a - \sqrt{a^2+4}$ must be negative, because both $-a$ and $-\sqrt{a^2+4}$ are both negative. Therefore, for this inequality to be true, we must have that $-a - \sqrt{a^2+4} < -2$, in order for it to satisfy the absolute value requirement. Earlier, we established that

$$-a - \sqrt{a^2+4} < -2$$

is a true statement for $a > 0$.

Therefore, for $a > 0$, taking $\pm \implies -$ in $\dfrac{-a \pm \sqrt{a^2+4}}{2}$ gives the stable fixed point of $f^{-1}$.

#### 5.2.2 When $a < 0$

For $a < 0$, setting the $\pm$ as $-$ yields the inequality

$$\left|-a - \sqrt{a^2+4}\right| > 2.$$

Setting $b = -a$, such that $b > 0$, and substituting gives us

$$\left|b - \sqrt{b^2+4}\right| > 2.$$

We have previously established that this statement is false for $b > 0$, and therefore, taking the $-$ from the $\pm$ yields a contradiction.

However, taking the $+$ from the $\pm$ gives

$$\left|-a + \sqrt{a^2+4}\right| > 2.$$

Substitution for $b = -a$, such that $b > 0$ gives

$$\left|b + \sqrt{b^2+4}\right| > 2.$$

We have previously shown this inequality is true for $b > 0$.

Therefore, for $a < 0$, taking $\pm \implies +$ in $\dfrac{-a \pm \sqrt{a^2+4}}{2}$ gives the stable fixed point of $f^{-1}$.

### 5.3 Conclusion

For $f(x)$, we see that the stable fixed point happens when we take the $+$ for $a > 0$ and the $-$ for $a < 0$. For $f^{-1}(x)$, we see that the stable fixed point arises from the $-$ for $a > 0$ and the $+$ when $a < 0$ yields the stable fixed point. We have thus managed to make the convergence formulas $\pm$-absent.

Furthermore, we have also proven that for the same value of $a$, convergence for infinite applications of $f$ and $f^{-1}$ take opposite signs of the $\pm$, and therefore the stable fixed point for one is the unstable fixed point of the other.

# 6 Further inquiry

The domain of this problem has many interesting directions for further inquiry. "6.4: Generalization to complex values of $a$" is the follow-up question for our paper.

## 6.1 The pattern of convergence for finite composition

In our project, we discussed infinite application of the function $f(x)$, as well as $f^{-1}(x)$. In building simulations to model the convergence, we ran into some interesting results. For instance, when $a = -0.25$, the oscillation pattern appears to be growing *larger*, as shown in Figure 2. However, after several iterations some value is reached such that the oscillating pattern *diminishes* in magnitude and eventually converges. This was of concern earlier in our exploration, because from a limited number of iterations (10) it seems that for certain values of $a$, $f^{\infty}(x)$ never converges. It would be interesting to find answers to two aspects of this phenomena:

1. For which values of $a$ does this pattern occur?
2. For what value of $f^n(x)$ does the oscillation pattern shift from growing larger to smaller?

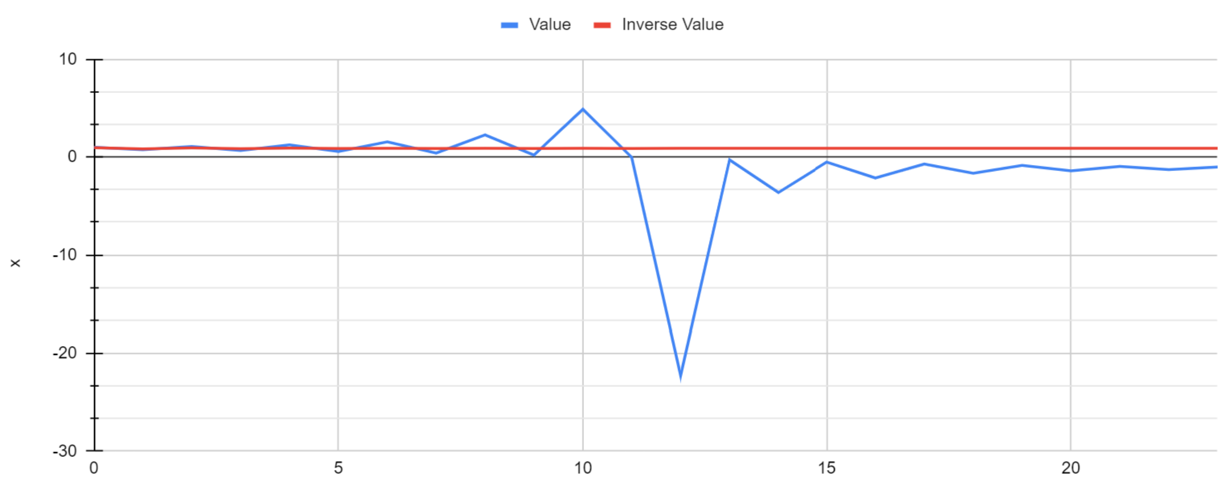


Figure 6.1: First 25 iterations when $a = -\frac{1}{4}$.

## 6.2 $f^a(x)$ approximates $a$ as $a$ takes on large values

It seems to be true that as $a$ increases, the continued fraction $a + \frac{1}{a+\frac{1}{a+\dots}}$ seems to give successively better approximations of $a$. That is, $\lim_{a\to\infty} a + \frac{1}{a+\frac{1}{a+\dots}} - a = 0$; or, in Gauss notation,

$$\lim_{k\to\infty} \left( \frac{\mathrm{K}_{a=1}^{k} \frac{1}{a}}{a} \right) = 1.$$

For very high values of $a$, the infinite composition converges to a value almost equal to $a$, and that the convergence seems to be closer to $a$ for higher values of it. See the table below for sample values. Another interesting direction of further inquiry would to find the nature of the difference between the approximation and 'real value', or $f^a(x) - a$.

| $a$ | **Approximate** $a+\frac{1}{a+\frac{1}{a+\dots}}$ **Convergence** |
|---|---|
| 1 | 1.618 |
| 2 | 2.414 |
| 5 | 5.193 |
| 50 | 50.019 |
| 100 | 100.009 |
| 1000 | 1000.001 |

## 6.3 When $a = 0$ for $\pm$-absent Convergence Formulas

When solving for the stable fixed point of $f(x)$ for $a = 0$, we are asked to find whether the inequalities:

$$\left|\frac{a+\sqrt{a^2+4}}{2}\right| > 1 \text{ and } \left|\frac{a-\sqrt{a^2+4}}{2}\right| > 1$$

are true. Since we know the value of $a$, we can plug it into the inequalities. We see that neither inequality is true for $a = 0$, as they both yield that $1 > 1$, which is not true. Thus, when $a = 0$ neither $+$ nor $-$ from the $\pm$ yields a stable fixed point.

This strange behavior is also exhibited in $f^{-1}(x)$ for $a = 0$, leading us to use more computation-based methods to determine the stable fixed points of $f(x)$ and $f^{-1}(x)$ for $a = 0$, as demonstrated below. Because both $f(x)$ and $f^{-1}(x)$ converge to the same point, somehow it must be both a stable and an unstable fixed point. A direction of further inquiry may be to make sense of these results.

### 6.3.1 For $f(x)$

We have that $f^{\infty}(x)$ can be written as

$$a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{\ddots}}}.$$

Setting $a = 0$ allows us to simplify the following expression to

$$0+\cfrac{1}{0+\cfrac{1}{0+\cfrac{1}{\ddots}}} = \cfrac{1}{\cfrac{1}{\cfrac{1}{\vdots}}} = 1.$$

Therefore, the stable fixed point of $f(x)$ for $a = 0$ is 1.

### 6.3.2 For $f^{-1}(x)$

We have that $f^{-\infty}(x)$ can be written as

$$\cfrac{1}{-a+\cfrac{1}{-a+\cfrac{1}{\ddots}}}.$$

Substituting $a = 0$ simplifies this expression to

$$\cfrac{1}{0+\cfrac{1}{0+\cfrac{1}{\ddots}}} = \cfrac{1}{\cfrac{1}{\cfrac{1}{\vdots}}} = 1.$$

This gives us that the stable fixed value of $f^{-1}(x)$ for $a = 0$ is 1.

## 6.4 Generalization to complex values of $a$

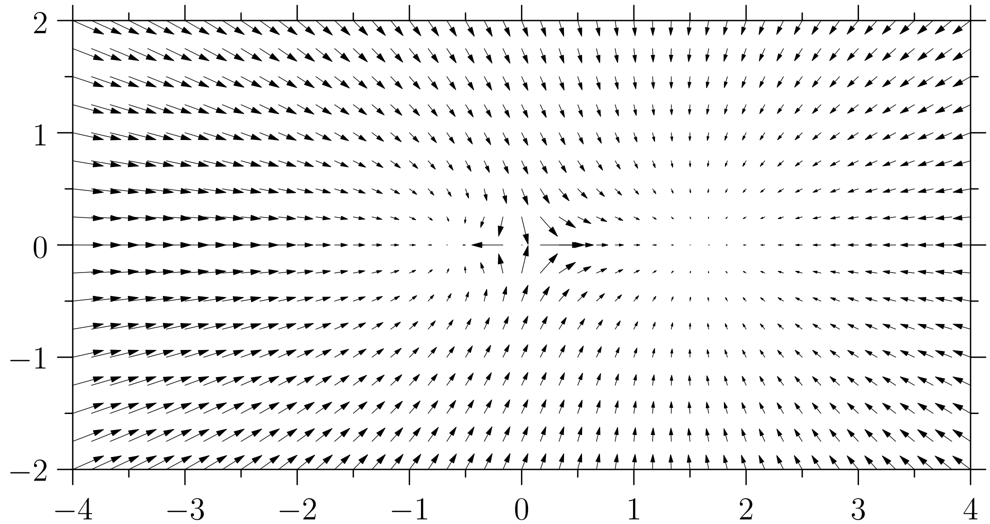


Figure 6.2

Arithmetic operations, such as addition and division, apply as soundly to complex numbers as they do to real numbers. In this follow-up section, we investigate where our earlier findings about convergence value generalize well to complex numbers.

We begin by finding convergence values for an arbitrary example, $a = 1 + i$, using the formula found before, then confirming if they are applicable with visualizations.

Setting $a = 1 + i$ in

$$x = \frac{a \pm \sqrt{a^2 + 4}}{2}$$

gives

$$x = \frac{1 + i \pm \sqrt{(1+i)^2 + 4}}{2} = \frac{1 + i \pm \sqrt{2i + 4}}{2}.$$

We can solve for $\sqrt{2i+4}$, by setting it to $a + bi$, giving the following system of equations:

$$a^2 - b^2 = 4$$

$$2abi = 2i$$

From the second equation, we have that $a = \frac{1}{b}$, and substituting into the first equation (using WolframAlpha) gives us the solution (although there are multiple, it doesn't matter if we take positive or negative solutions, because the $\pm$ varies the sign anyways) of

$$(a, b) = \left(\sqrt{2 + \sqrt{5}}, \sqrt{-2 + \sqrt{5}}\right) \implies \sqrt{2i + 4} = \sqrt{2 + \sqrt{5}} + i\sqrt{-2 + \sqrt{5}}.$$

Therefore, the solutions for $x$ are

$$\frac{1 + \sqrt{2 + \sqrt{5}}}{2} + i\frac{1 + \sqrt{-2 + \sqrt{5}}}{2} \text{ and } \frac{1 - \sqrt{2 + \sqrt{5}}}{2} + i\frac{1 - \sqrt{-2 + \sqrt{5}}}{2}$$

The decimal approximations for these are

$$1.52908 + 0.74293i \text{ and } -0.52908 + 0.25706i$$

These two fixed points are present in the complex plane, as shown below in Figure 6.6. Furthermore, we observe that one of the convergence solutions is *stable*, and the other is *unstable*. This is demonstrated by Figure 6.7, which visualizes the position of points over iterations converging to the stable fixed point.

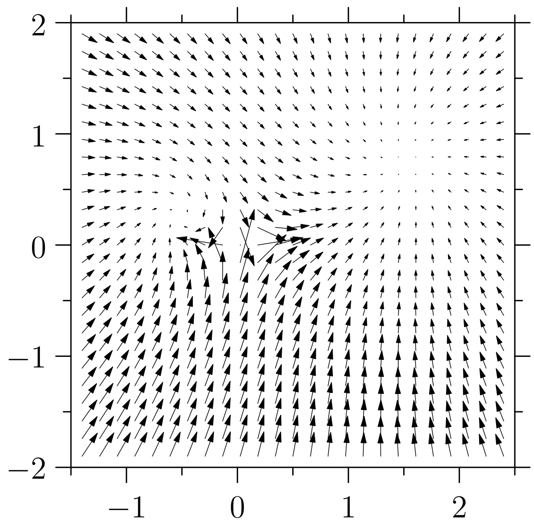


Figure 6.3: Resulting vector field for $f(x)$ when $a = 1+i$. We observe two fixed points, one at $-0.53 + 0.26i$ and the other at $1.53 + 0.74i$. Erratic behavior occurs around the origin due to division by zero.

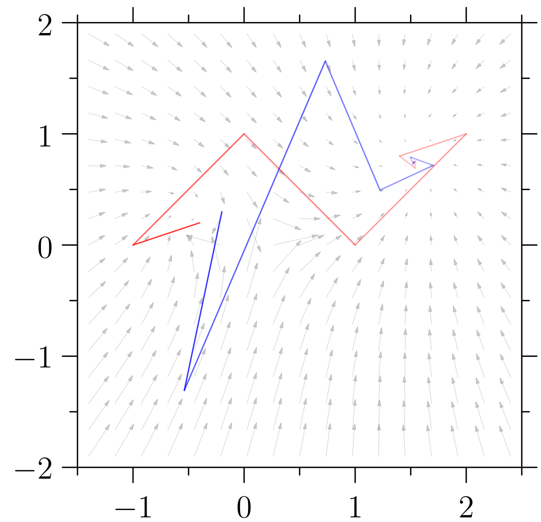


Figure 6.4: Iterating $f$ from starting values of $-0.4+0.2i$ and $-0.2+0.3i$, we observe that they both converge to the fixed point on the right — which is the stable fixed point of the two.

For another example, consider $a = \frac{3}{4} - \frac{2}{3}i$.
Plugging $\frac{3}{4} - \frac{2}{3}i$ into the formula, we get

$$\frac{\frac{3}{4} - \frac{2}{3}i \pm \sqrt{\left(\frac{3}{4} - \frac{2}{3}i\right)^2 + 4}}{2} = \frac{\frac{3}{4} - \frac{2}{3}i \pm \sqrt{\frac{593}{144} + i}}{2}$$

To evaluate the $\sqrt{\frac{593}{144} + i}$, we set the answer to $a+bi$, and notice that $(a+bi)^2 = a^2 - b^2 + 2abi$. Therefore, as in the previous part,

$$a^2 - b^2 = \frac{593}{144}$$
$$2ab = 1$$

From the second equation, we have that $a = \frac{1}{b}$. Substituting this into the first equation, then solving, we get that $\sqrt{\frac{593}{144} + i} \approx 2.044 + 0.245i$, and therefore that

$$\frac{\frac{3}{4} - \frac{2}{3}i + \sqrt{\frac{593}{144} + i}}{2} = 1.397 + 0.456i$$

$$\frac{\frac{3}{4} - \frac{2}{3}i - \sqrt{\frac{593}{144} + i}}{2} = -0.647 + 0.211i$$

Thus, $a = \frac{3}{4} - \frac{2}{3}i$ goes to $1.397 + 0.456i$ and $-0.647 + 0.211i$. Looking at the graph below, we see that these two points are indeed the fixed points of the graph. The same behavior with stable and unstable fixed points has been observed; we have found $1.397 + 0.456i$ is the stable fixed point. Although this paper does not prove it, we hypothesize that like how $|b| > 1$ (where $b$ was a fixed point) for it to be stable for real numbers, *the magnitude of a complex number must be larger than 1* for it to be stable.

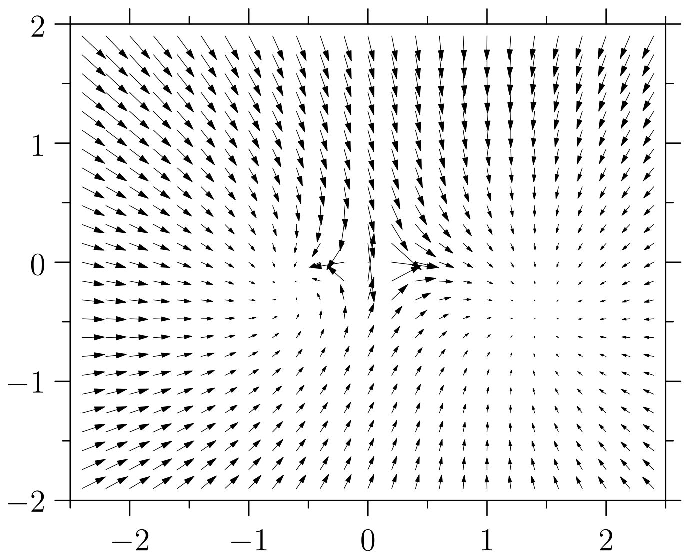


Figure 6.5: Resulting vector field for $f(x)$ when $a = \frac{3}{4} - \frac{2}{3}i$

Thus, the formula derived — $x = \frac{a \pm \sqrt{a^2+4}}{2}$ — does work on complex numbers, as confirmed through visualization and calculation (albeit not rigorously, as complex analysis is currently out of the reach of us). Further inquiry on this follow-up would prove if our hypothesis relating the stability of complex numbers with magnitudes larger than 1 was true or not.