# Intersections of Constricted Lines Within a Unit Circle

Andre Ye

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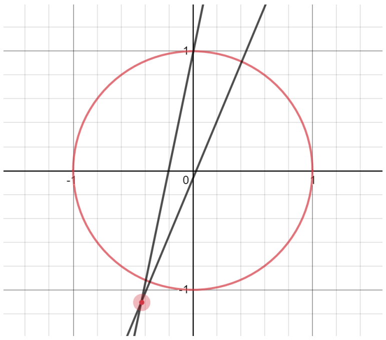
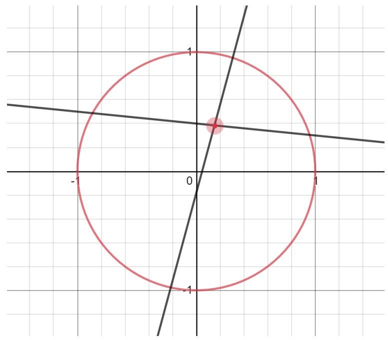
# 1 Introduction

## 1.1 The Problem and Deriving the Inequality

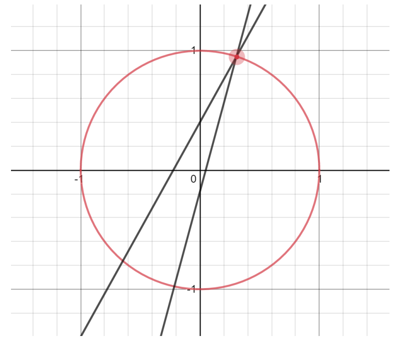
Consider two lines, written as $y = m_1x + b_1$ and $y = m_2x + b_2$; their $y$-intercepts must be between $-1$ and 1, inclusive (that is, $-1 \le b_1 \le 1$ and $-1 \le b_2 \le 1$). Thus, their intersection may fall inside, outside, or on the unit circle, defined by the equation $x^2 + y^2 = 1$. Alternatively, the two lines may not meet at all, or may meet 'everywhere' (being coincident); these are two cases explored in Section 2.1: Parallel and Coincident Lines.



(a) The intersection of $y = 2.4x - 0.07$ and $y = 4.83x + 0.975$. The point lands outside of the unit circle boundary.



(b) The intersection of $y = -0.1x + 0.4$ and $y = 3.7x - 0.18$. The point lands inside the unit circle boundary.



(c) The intersection of $y = 1.8x + 0.401$ and $y = 3.7x - 0.18$. The point lands approximately on unit circle boundary.

Figure 1: Examples of two lines and their intersections.

It's important to note that because the $y$-intercepts must be between $-1$ and 1 inclusive, each of the lines must touch the unit circle at least once.

We would like to understand, first, what the requirements are for the two lines satisfying the $y$-intercept constraints to intersect inside the unit circle. We can find the point representing the intersection of these two lines as follows:

$$\begin{aligned} m_1x + b_1 &= m_2x + b_2 \\ m_1x - m_2x &= b_2 - b_1 \\ (m_1 - m_2)x &= b_2 - b_1 \\ x &= \frac{b_2 - b_1}{m_1 - m_2} \end{aligned}$$

Plugging in this derived value of $x$ for $y$ and simplifying:

$$\begin{aligned} y &= m_1\left(\frac{b_2 - b_1}{m_1 - m_2}\right) + b_1 \\ &= \frac{m_1b_2 - m_1b_1}{m_1 - m_2} + b_1 \\ &= \frac{m_1b_2 - m_1b_1}{m_1 - m_2} + \frac{b_1m_1 - b_1m_2}{m_1 - m_2} \\ &= \frac{m_1b_2 - m_1b_1 + b_1m_1 - b_1m_2}{m_1 - m_2} \\ &= \frac{m_1b_2 - m_2b_1}{m_1 - m_2} \end{aligned}$$

Hence, the point of intersection is $\left(\frac{b_2-b_1}{m_1-m_2}, \frac{m_1b_2-m_2b_1}{m_1-m_2}\right)$. To solve for when these values are *less* than 1 unit away from the origin, we can write an inequality:

$$\left(\frac{b_2 - b_1}{m_1 - m_2}\right)^2 + \left(\frac{m_1 b_2 - m_2 b_1}{m_1 - m_2}\right)^2 < 1$$
$$\frac{(b_2 - b_1)^2 + (m_1 b_2 - m_2 b_1)^2}{(m_1 - m_2)^2} < 1$$
$$(b_2 - b_1)^2 + (m_1 b_2 - m_2 b_1)^2 < (m_1 - m_2)^2$$

Because the differences are squared, the order does not matter; thus the inequality can be rewritten as $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 < (m_1 - m_2)^2$. The test to determine whether the equations lie inside the unit circle or not is if the differences in $y$-intercept squared added to the difference in cross-multiplying slope and $y$-intercept squared is less than the difference in slope squared.

This inequality elegantly connects the nature of lines with that of circles and hence there is, along for many other interesting quirks, interest in understanding more about it.

### 1.2 Derived Questions

Naturally, as with any meaningful inequality, questions arise as to how close or far apart the two quantities - the left-hand side (LHS) and right-hand side (RHS) - can be. Thus, in exploring how to better understand this inequality, we can ask:

- What are the maximum and minimum possible values achieved by the LHS and the RHS (irrespective of the other)? Are these quantities bounded? What is the graph of lines that achieve these values?
- What is the minimum and maximum possible discrepancy between the LHS and RHS while the intersection still resides in the circle? What is the graph of these lines, and where does their intersection reside? How can lines that form the maximum and minimum possible discrepancy be formed?

These questions allow us to understand the nature of circles from the perspectives of lines, being two distinct building blocks of algebra and geometry.

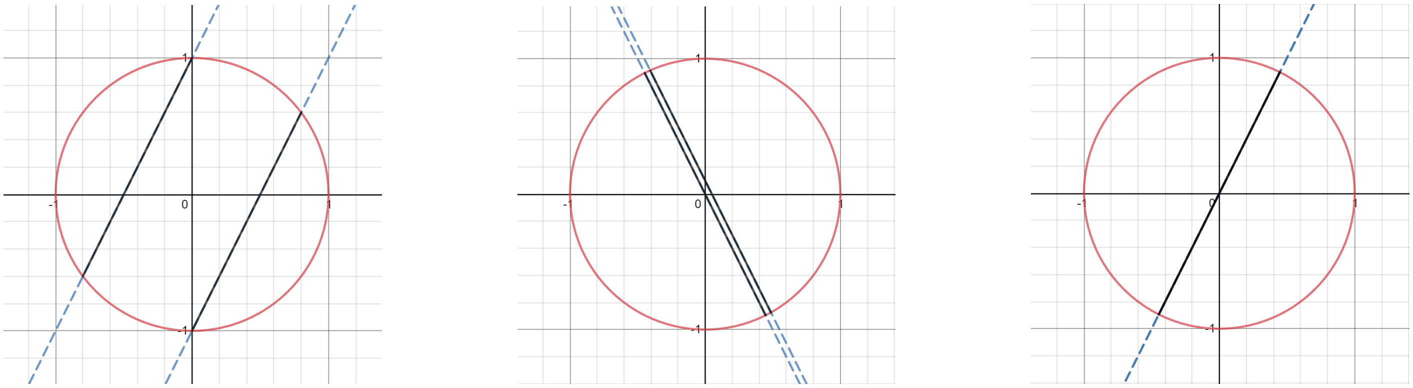
## 2 Bounds of the Inequality

We would like to understand the bounds of the the LHS and the RHS of the inequality $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 < (m_1 - m_2)^2$, both in special cases of parallel and coincident lines, as well as general lines.

### 2.1 Parallel and Coincident Lines

If two lines are parallel, they will never intersect, inside or outside of the unit circle. If two lines are parallel, they will have the same slopes and different $y$-intercepts; thus $m_1 = m_2$ and $b_1 \neq b_2$. The RHS, $(m_1 - m_2)^2$, evaluates to zero. The LHS evaluates to $(m_1 b_2 - m_2 b_1)^2 = (m_1 b_2 - m_1 b_1)^2 = (m_1 b_1 - m_1 b_2)^2 = (m_1(b_1 - b_2))^2 = m_1^2 \cdot (b_1 - b_2)^2$. ($m_1$ is an arbitrary choice, it can be substituted with $m_2$ since the two are equivalent.) The maximum LHS can be obtained by setting $m_1 = m_2 = \infty$ or $m_1 = m_2 = -\infty$ and setting $b_1 = -1$ and $b_2 = 1$ (or vice versa). This yields $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 = 4 + (m_1(b_2 - b_1))^2 = 4 + (2m_1)^2 = 4m_1^2 + 4$. Note that using the notation "variable $= \infty$" simply means "setting the variable to an arbitrarily large number". In this case, the LHS is as far from the RHS as one can get; thus the inequality is 'at its most unequal'. The LHS is *minimized*, however, as the slopes and $y$-values near each other. When $m_1 = m_2$ and $b_1 = b_2$, the two lines occupy the same space and intersect at *every single point*.

Coincident lines occur when the slopes and $y$-intercepts are equal ($m_1 = m_2$ and $b_1 = b_2$). The RHS evaluates to zero, as $(m_1 - m_2)^2 = (m_1 - m_1)^2 = 0$, and the LHS also can be reduced to zero, as $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 = (b_1 - b_1)^2 + (m_1 b_1 - m_1 b_1)^2 = 0$. Thus, the inequality becomes $0 < 0$, which is *just barely* on the edge of satisfying the inequality. This is a satisfying solution; coincident lines intersect everywhere, so they are very close to intersecting 'within the circle' but *don't quite count*. Since this section considers lines with the same slope ($m_1 = m_2$), the RHS will always evaluate to 0; therefore it is impossible to satisfy the inequality, but it is possible to get infinitely close when the two lines are coincident.



(a) Lines $y = 2x + 1$ and $y = 2x - 1$. LHS is $(2)^2 + (2(1) - 2(-1))^2 = 20$.

(b) Lines $y = -2x$ and $y = -2x + 0.1$. LHS is $(0.1)^2 + (-2(0) - (-2)(0.1))^2 = 0.05$.

(c) Lines $y = 2x$ and $y = 2x$. LHS is 0.

Figure 2: LHS for lines with the same slope.

### 2.2 General Lines

The RHS, $(m_1 - m_2)^2$, can take up any value in the range $[0, \infty)$. On the other hand, the maximum value of the $y$-intercept component of the LHS, $(b_1 - b_2)^2$, is 4 (with $b_1 = 1$ and $b_2 = -1$). The other component, $(m_1 b_2 - m_2 b_1)^2$, can take up any value in the range $[0, \infty)$. $b_1$ and $b_2$, restrained by $[-1, 1]$ act like coefficients in a linear combination.

## 3 Maximum and Minimum Inequality Discrepancies

In this section, we seek to understand the maximum and minimum possible differences between the LHS and RHS of the inequality $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 < (m_1 - m_2)^2$.

### 3.1 Establishing Standards and Definitions of Inequality Discrepancies

What a 'minimum discrepancy' or 'maximum discrepancy' is fairly vague at this point. For one, one could object that the inequality could just as naturally be represented by $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 < (m_1 - m_2)^2$ as by $(m_1 - m_2)^2 - (m_1 b_2 - m_2 b_1)^2 > (b_1 - b_2)^2$.

For the purposes of this paper, the inequality used will be $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 < (m_1 - m_2)^2$, for a few reasons. Firstly, it is the form the inequality was in when it was derived. Secondly, when the inequality is in this form it resembles the form of a circle, which we have established in section 1.2: The Inequality is a Circle in Disguise to be elegantly connected to the format of a circle with no additive or subtractive algebraic manipulations.

Thus, it follows that the LHS and RHS, as they have been referred to previously in the paper, are $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2$ and $(m_1 - m_2)^2$, respectively. The 'maximum inequality discrepancy' refers to the largest possible difference that can be obtained between the LHS and the RHS while the inequality is still true. On the other hand, the 'minimum inequality discrepancy' refers to the smallest possible difference that can be obtained between the LHS and the RHS while still satisfying the inequality.

The inequality given determines whether a point of intersection falls inside or outside a unit circle. The difference between the components of the inequality should also indicate *how close a point is* to falling inside or outside a circle. Thus, two lines that produce a large discrepancy should be 'the most inside the circle' and two lines that produce the least discrepancy should be 'the least inside the circle'. These ideas will be further explored in the following sections.

The variable $a$ will be used to indicate an 'arbitrarily large (positive) number'. However, its value will be constant and it will be used in comparisons of expressions. Even though, for example, $16a^3$ and $2a$ both involve an arbitrarily large number, we can say that $16a^3 > 2a$. Similarly, $\frac{1}{a}$ indicates 'arbitrarily small number', 'small' in the sense of 'close to zero' rather than $-a$.

## 3.2 Maximum Inequality Discrepancy

One conjecture, which we will name the 'Origin Intersection Conjecture' proposes that lines with intersections at the origin (0,0) produce the smallest LHS. The rationale behind this conjecture is as follows: as the largest discrepancy is the point that is 'most inside the circle', the origin seems like a natural point, as it is the inner-most point of a circle. More formally, we may state the conjecture as: the largest possible discrepancy between the RHS and the LHS of the inequality occurs only when the intersection of the two lines in question is at the origin.

To evaluate this, consider that for the intersection of the two lines to be at the origin, $b_1 = b_2 = 0$. Thus, the inequality can be simplified as such:

$$\begin{aligned} (b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 &< (m_1 - m_2)^2 \\ (0)^2 + (m_1(0) - m_2(0)) &< (m_1 - m_2)^2 \\ 0 &< (m_1 - m_2)^2 \end{aligned}$$

By restricting lines to pass through the origin, it certainly becomes simple to understand the relationship between lines. The maximum difference can be achieved when $m_1 = a$ and $m_2 = -a$, where $a$ is an arbitrarily large number. The RHS of the simplified inequality then becomes $(a - (-a))^2 = 4a^2$. Thus, it is guaranteed for two lines that both pass through the origin to satisfy the inequality unless they are coincident, in which a $0 < 0$ scenario as discussed in Section 2.1: Parallel and Coincident lines arises. Thus, if the Circle-Rim Intersection Conjecture is true, then the maximum possible discrepancy between the LHS and RHS is given by $|\text{RHS} - \text{LHS}| = |4a^2 - 0| = 4a^2$. On the other hand, having another set of slopes as 0 and $a$ yields only $a^2$, four times less than if the slopes are chosen to be negatives of each other.

It's important to note that, when describing discrepancies, we are not comparing the actual possible discrepancy - arbitrarily large quantities cannot be compared in this fashion - but instead the *impact* changing one or many of the 'four maneuvers', two slopes and two $y$-intercepts, have on the discrepancy. For example, both $4a^2$ and $a^2$ can be thought of conceptually as $\infty$ in 'value', but manipulating the two slopes to be $-a$ and $a$ has a higher impact than using 0 and $a$. One can think of measuring impact as 'how much does the quantity change if we shift $a$ by some value?' This is what makes the comparison of technically arbitrarily large or small quantities meaningful.

One can prove the Origin Intersection Conjecture premised on one fact: the minimum value of the LHS cannot be negative, given the the $y$-intercepts are real numbers between $[-1, 1]$. If the $y$-intercepts are not both equal to 0, the LHS must become larger than zero, unless the two lines are coincident. In this case, however, as discussed above, the RHS is also zero, and therefore the inequality is not satisfied. Therefore, the minimum possible value of the LHS is 0; as the maximum possible value of the RHS is $4a^2$ (as discussed in the paragraph above).

The Origin Intersection Conjecture is true - the two lines that yield the largest inequality discrepancy intersect at the origin. Furthermore, however, it shows the dynamics of slope in the calculation. Even as it is guaranteed that the intersection will fall inside a circle, the inequality rewards higher discrepancies (higher 'confidence') to lines with steeply positive and negative slopes. On the other hand, the inequality is much less 'confident' and barely satisfied if the two slopes were to be, for example, $m_1 = 1$ and $m_2 = 1 + \frac{1}{a}$.

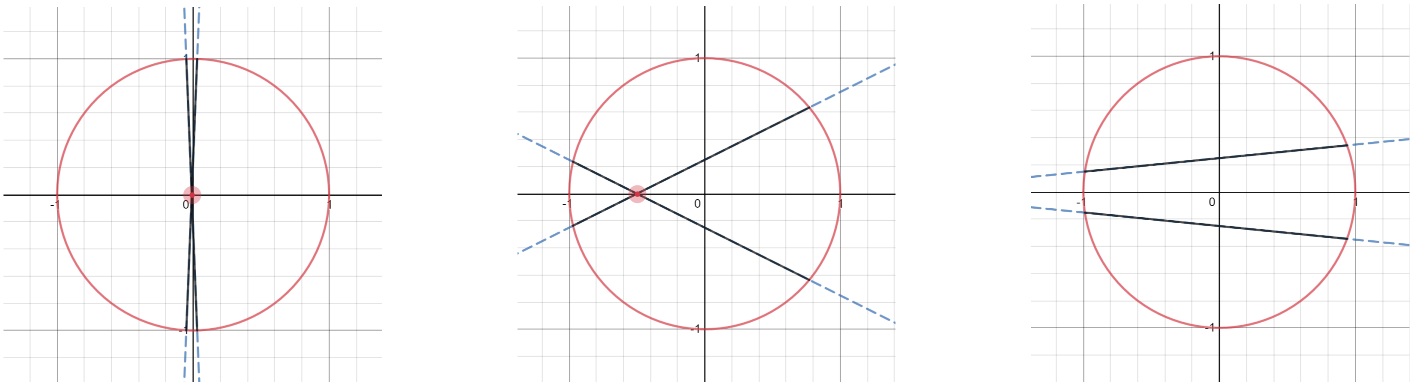
This makes sense, graphically, and allows us to understand the inequality past lines restricted to pass through the origin. If two lines are working towards each other - one sufficiently large positive slope and one sufficiently large negative slope - they are likely (but not guaranteed) to meet inside the unit circle. The inequality 'rewards' lines that more steeply approach each other. This is true for all cases of lines in the outlined problem. For example, consider the graphs of lines with varying values of slope and $y$-intercepts of 0.25 and $-0.25$, visualized below in Figure 3.

With a sufficiently 'polarized' set of slopes ('polarized' being loosely defined as lines that point against each other), the position of the intersection of the two lines nears that of the origin, regardless of the $y$-intercepts: an elegant circular connection.

Given this, the graph of two lines with the largest possible discrepancy between the LHS and the RHS while still satisfying the inequality looks somewhat like the below visualization (with liberties taken for visualization - the actual graph of largest discrepancy would consist of two almost vertical lines).

## 3.3 Minimum Inequality Discrepancy and the Circle-Rim Intersection Conjecture

A *related solution* to this question of minimum discrepancy, as discussed in section 2.1: Parallel and Coincident Lines, is the intersection formed by two coincident lines. However, per defined in section 3.1: Establishing Standards and Definitions

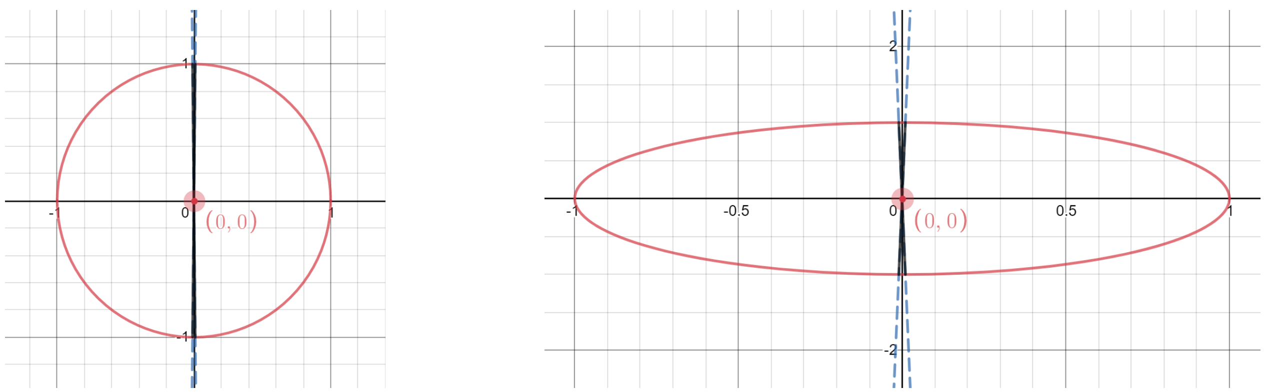


(a) Lines $y = 25x + 0.25$ and $y = -25x - 0.25$. Intersection is within the circle.

(b) Lines $y = 0.5x + 0.25$ and $y = -0.5x - 0.25$. Intersection is within the circle.

(c) Lines $y = 0.1x + 0.25$ and $y = -0.1x - 0.25$. Intersection is outside the circle.

Figure 3: Demonstration of why polarized slopes are 'rewarded' with larger inequality discrepancies.




(a) Normal view.

(b) Roughly 1:4 $x$-axis to $y$-axis view.

Figure 4: Display of lines satisfying the maximum discrepancy.

of Inequality Discrepancies, a discrepancy can only be observed when the *inequality is still satisfied.* One way to *arrive* at a minimum discrepancy is to first find lines that satisfy the 'border condition' (e.g. $0 < 0$), then to make some infinitely small change to the lines in question to tip them just enough to satisfy the inequality (e.g. $0 - \frac{1}{a} < 0$).

Coincident lines can be handled in this way. They achieve the border condition $0 < 0$ in that they are just on the border of satisfying the inequality. Making one specific small changes to the four mechanisms of change - being the two $y$-intercepts and two slopes - can tip the inequality to be true.

- Changing the $y$-intercepts converts coincident lines to parallel lines in that the $y$-intercepts are different but the slopes remain the same, which do not satisfy the inequality (as discussed in 2.1: Parallel and Coincident Lines).
- Changing the slopes while keeping both the $y$-intercepts in the equation $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 < (m_1 - m_2)^2$ yields $0 + (b_1(m_1 - m_2))^2 < (m_1 - m_2)^2 \rightarrow \left(b_1\left(\frac{1}{a}\right)\right)^2 < \left(\frac{1}{a}\right)^2 \rightarrow |b_1| < 1$. Therefore, as long as the intercepts are not 1 or $-1$, this strategy of beginning with two coincident lines and nudging one by a small amount satisfies the inequality. The discrepancy is RHS – LHS $= \left(\frac{1}{a}\right)^2 - \left(\frac{b_1}{a}\right)^2$. When $b_1 = 0$, the expression evaluates to $\frac{1}{a^2}$. Note that we choose $b_1$ arbitrarily, since it is equal in value to $b_2$ this is not important. We represent $m_1 - m_2$ as $\frac{1}{a}$, since the difference between the two should be infinitesimal.
- Changing the slope a minimal amount in addition to the $y$-intercept, as explored in the previous section 3.2:

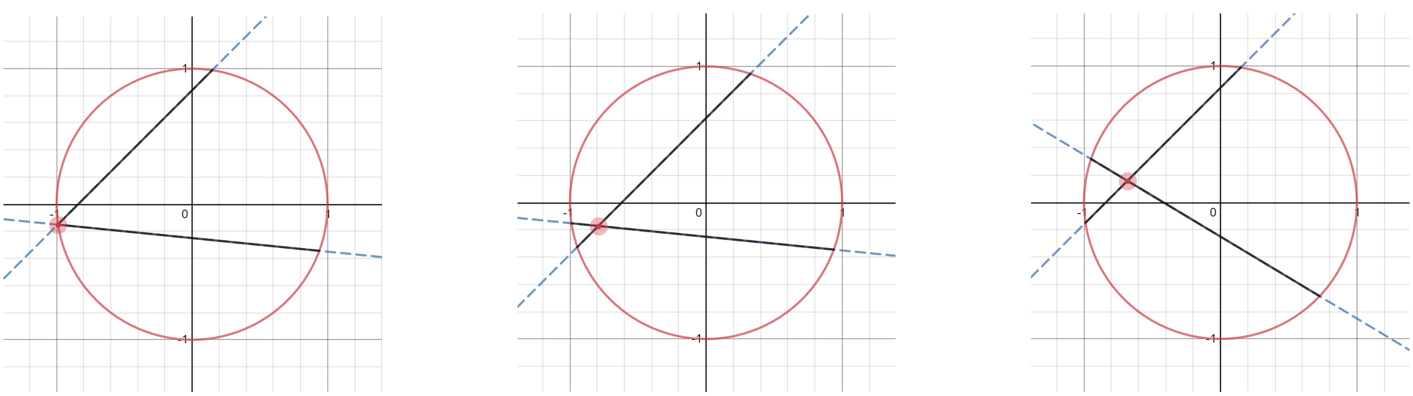
Maximum Inequality Discrepancy, makes it more difficult to satisfy the inequality. For the inequality to be satisfied with differing $y$-intercepts, a major change needs to be made to the slope (exemplified by plots in Figure 3, in which large changes need to be made to the slopes of the two lines in order for the intersection to fall inside the circle). In this case, coincident lines can no longer be considered to be a boundary case, since the change made is significant.

Therefore, it is possible for any two lines, as long as they share the same $y$-intercept, to have a minimum discrepancy of $\frac{1}{a^2}$ if their slopes differ by $\frac{1}{a}$. However, there may be other cases that are able to achieve minimum discrepancies.

One conjecture - the 'Circle-Rim Intersection Conjecture' - states that lines that intersect on the perimeter of the circle produce the smallest discrepancy. The rationale behind this conjecture is as follows: as the smallest discrepancy is the point that is the 'least inside the circle' while still being 'in' the circle, then the rim of the circle seems like a natural point, for it is the inner-most point of a circle. More formally, we may state the conjecture as: the smallest possible discrepancy between the RHS and the LHS of the inequality occurs only when the intersection of the two lines in question occurs on the perimeter of the circle.

With a little thought, the Circle-Rim Intersection Conjecture seems to be quite obviously true, and by definition. Given the previously discussed strategy of finding the border case, it was proved in Section 1.2: The Inequality is a Circle in Disguise that $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 = (m_1 - m_2)^2$ was the equation for a circle. By definition, this is as close as the LHS and the RHS can be from each other (they are equal, their difference is 0). However, per defined in Section 3.1: Establishing Standards and Definitions of Inequality Discrepancies, the two lines still need to satisfy the condition, whereas $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 = (m_1 - m_2)^2$ is *on* the circle but not *inside* the circle. Equally important to finding that the smallest discrepancy possible can be achieved when the intersection is on the intersection is finding how to construct such a pair of lines. We will do this by finding how to make some infinitely small change - which will occur in the form of some combination of the four maneuvers available to us - changes to $m_1$, $m_2$, $b_1$, $b_2$ - to tip the inequality to become true.

This is a meaningful question, because it is not immediately apparent how these mechanisms apply. For instance, consider the following set of lines whose intersection lines on the rim of the circle (Figure 5, left). In the case visualized, adjusting the $y$-intercept and the slope to be smaller works, but can this be generalized? Is it possible that there are sets of lines like the coincident lines discussed above where changing the slope or changing the $y$-intercept in a particular direction, or at all, do *not* tilt the inequality to become true?



(a) Original lines with intersection on the circle.

(b) Decreased $y$-intercept of one line; intersection now resides inside circle.

(c) Decreased the slope of another line (made it more steep negatively); intersection now resides inside circle.

Figure 5: Demonstration of the impact of 'nudges' on different variables

There are two avenues to answering this question: making changes to the slope and making changes to the $y$-intercept. In each, we will find how to shift the intersection to land just inside the circle.

### 3.3.1 Nudges to the Slope

Let us consider the change to one slope; thus we can write $m_1$ as $m_1 + j$, where $j$ is some sort of small 'nudge' (change) applied to the slope. Hence, $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 = (m_1 - m_2)^2$ can be rewritten as:

$$(b_1 - b_2)^2 + ((m_1 + j)\, b_2 - m_2 b_1)^2 = ((m_1 + j) - m_2)^2$$

We are interested, primarily, in the *difference* such a nudge makes; this will entail a (very messy but worthwhile) expansion. First, we can find the impact a nudge makes on the RHS:

$$\begin{aligned}
\text{Nudged value} - \text{Original value} &= \left((b_1 - b_2)^2 + ((m_1 + j)\, b_2 - m_2 b_1)^2\right) - \left((b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2\right) \\
&= b_1^2 - 2b_1 b_2 + b_2^2 + b_2^2 (m_1 + j)^2 - 2b_1 b_2 m_2 (m_1 + j) + b_1^2 m_2^2 - \left((b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2\right) \\
&= b_1^2 - 2b_1 b_2 + b_2^2 + b_2^2 m_1^2 + 2jb_2^2 m_1 + j^2 b_2^2 - 2b_1 b_2 m_2 (m_1 + j) \\
&\quad + b_1^2 m_2^2 - \left((b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2\right) \\
&= b_1^2 - 2b_1 b_2 + b_2^2 + b_2^2 m_1^2 + 2jb_2^2 m_1 + j^2 b_2^2 - 2b_1 b_2 m_1 m_2 \\
&\quad - 2jb_1 b_2 m_2 + b_1^2 m_2^2 - (b_1 - b_2)^2 - (m_1 b_2 - m_2 b_1)^2 \\
&= j^2 b_2^2 + 2jb_2^2 m_1 - 2jb_1 b_2 m_2
\end{aligned}$$

Secondly, we can find the impact of such a nudge on the LHS:

$$\begin{aligned}
\text{Nudged value} - \text{Original value} &= \left(((m_1 + j) - m_2)^2\right) - \left((m_1 - m_2)^2\right) \\
&= -2m_1 m_2 + 2m_1 j + m_1^2 + m_2^2 + j^2 - 2m_2 j - \left(m_1^2 - 2m_1 m_2 + m_2^2\right) \\
&= -2m_1 m_2 + 2m_1 j + m_1^2 + m_2^2 + j^2 - 2m_2 j - m_1^2 + 2m_1 m_2 - m_2^2 \\
&= 2m_1 j + j^2 - 2m_2 j
\end{aligned}$$

Because we are beginning with the 'border case' RHS = LHS and seeking to find the impact of small changes form that point, we only need to be concerned with Nudge to RHS and Nudge to LHS. More specifically, however, we would like to understand scenarios where the nudge to LHS is *just barely* less than the nudge to the RHS to satisfy the inequality, something like Nudge to LHS < Nudge to RHS. This can also be expressed as Nudge to LHS − Nudge to RHS < 0.

$$\begin{aligned}
\text{Nudge to LHS} - \text{Nudge to RHS} &< 0 \\
\left(2m_1 j + j^2 - 2m_2 j\right) - \left(j^2 b_2^2 + 2jb_2^2 m_1 - 2jb_1 b_2 m_2\right) &< 0 \\
2m_1 j + j^2 - 2m_2 j - j^2 b_2^2 - 2jb_2^2 m_1 + 2jb_1 b_2 m_2 &< 0 \\
\left(1 - b_2^2\right) j^2 + (2m_1 - 2m_2 - 2b_2^2 m_1 + 2b_1 b_2 m_2) j &< 0
\end{aligned}$$

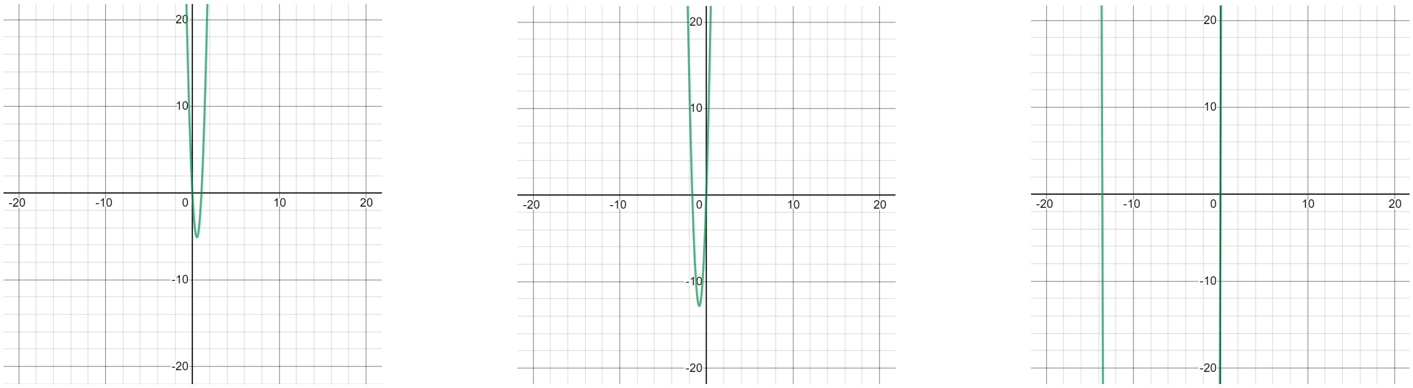
This representation is important; a visual analysis (demonstrated in Figure 6) will be important for the following steps. Two key observations include a) the quadratic always passes through the origin, and b) the quadratic always points up.

To solve for $j$, we can use the quadratic formula to find when this quadratic is equal to 0, then use the two roots as non-inclusive bounds for possible values of $j$.

$$j = \frac{-\left(2m_1 - 2m_2 - 2b_2^2 m_1 + 2b_1 b_2 m_2\right) \pm \sqrt{\left(2m_1 - 2m_2 - 2b_2^2 m_1 + 2b_1 b_2 m_2\right)^2 - 0}}{2\,(1 - b_2^2)}$$

Solving for one case:

$$\begin{aligned}
&\frac{-\left(2m_1 - 2m_2 - 2b_2^2 m_1 + 2b_1 b_2 m_2\right) + \sqrt{\left(2m_1 - 2m_2 - 2b_2^2 m_1 + 2b_1 b_2 m_2\right)^2 - 0}}{2\,(1 - b_2^2)} \\
&= \frac{-\,(2m_1 - 2m_1 b_2 + 2m_2 b_2 b_1 - 2m_2) + 2m_1 - 2m_1 b_2 + 2m_2 b_2 b_1 - 2m_2}{2\,(-b_2 + 1)} \\
&= \frac{0}{2\,(1 - b_2)} \\
&= 0
\end{aligned}$$



(a)

(b)

(c) This quadratic's vertex is below the camera line in this image.

Figure 6: Graphs of the difference between the RHS nudge and the LHS nudge for different values of $m_1$, $m_2$, $b_1$, and $b_2$, with $j$ as the $x$ axis and the output of the expression as the $y$ axis.

As demonstrated by the graph, one bound is $j = 0$. Solving for another case:

$$\begin{aligned}
&\frac{-\left(2m_1 - 2m_2 - 2b_2^2 m_1 + 2b_1 b_2 m_2\right) - \sqrt{\left(2m_1 - 2m_2 - 2b_2^2 m_1 + 2b_1 b_2 m_2\right)^2 - 0}}{2\left(1 - b_2^2\right)} \\
&= \frac{-2\left(2m_1 - 2m_2 - 2b_2^2 m_1 + 2b_1 b_2 m_2\right)}{2\left(1 - b_2^2\right)} \\
&= -\frac{2m_1 - 2m_2 - 2b_2^2 m_1 + 2b_1 b_2 m_2}{1 - b_2^2}
\end{aligned}$$

Because the graph orients up (and must do so because $1 - b_2$, the coefficient for $j^2$, cannot be negative), values of $j$ that produce negative differences - ones that satisfy the inequality - are in the range $(0, -\frac{2m_1 - 2m_2 - 2b_2^2 m_1 + 2b_1 b_2 m_2}{1 - b_2^2})$, noting that it may need to be rewritten as $(-\frac{2m_1 - 2m_2 - 2b_2^2 m_1 + 2b_1 b_2 m_2}{1 - b_2^2}, 0)$ if $-\frac{2m_1 - 2m_2 - 2b_2^2 m_1 + 2b_1 b_2 m_2}{1 - b_2^2}$ is less than 0. As a re clarification: we solved for values of $j$ in which the nudge to the LHS was less than the nudge to the RHS; by choosing values of $j$ near the roots but still satisfy the LHS $<$ RHS condition, we can make the LHS nudge as close to the RHS nudge as possible. Looking at the bound of 0, we can determine a process for making nudges to a slope to just barely satisfy the inequality:

1. Find the value of $-\frac{2m_1 - 2m_2 - 2b_2^2 m_1 + 2b_1 b_2 m_2}{1 - b_2^2}$.
2. If that value is larger than 0, make a positive infinitesimally small change to $m_1$.
3. If that value is less than 0, make a negative infinitesimally small change to $m_1$.

These judgements on whether to add or subtract infinitely small amounts can be further demonstrated visually. Consider graphs a and b in Figure 6.

- Figure 6a has a positive range of values of $j$ in which the result is (as desired) negative. For the smallest possible change, we choose the closest positive value of $j$ to 0 that is not 0, since choosing $j = 0$ would have no impact on tilting the inequality towards being true.
- Figure 6b has a negative range of values of $j$ in which the result is negative. For the smallest possible change, we choose the closest negative value of $j$ to 0 that is not 0.

### 3.3.2 Nudges to the $y$-intercept

We have established that it is possible, and outlined a method, to construct two lines that satisfy the minimum possible discrepancy, by adjusting the slope. Let us consider a similar exploration by adjusting the $y$-intercept, in which $b_1$ is replaced with $b_1 + k$, where $k$ is some sort of small nudge applied to $b_1$. Hence, $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 = (m_1 - m_2)^2$ can be rewritten as:

$$((b_1 + k) - b_2)^2 + (m_1 b_2 - m_2 (b_1 + k))^2$$

As mentioned above, we are interested in the difference in value the nudge has. For the RHS, conveniently $y$-intercepts are not involved and the difference is 0. The LHS difference can be found as follows:

$$\begin{aligned}
&\left((b_1 + k - b_2)^2 + (m_1 b_2 - m_2 (b_1 + k))^2\right) - \left((b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2\right) \\
&= (b_1 + k - b_2)(b_1 + k - b_2) + b_2^2 m_1^2 - 2 b_2 m_1 m_2 (b_1 + k) + m_2^2 (b_1 + k)^2 - \left((b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2\right) \\
&= 2kb_1 - 2b_1 b_2 + b_1^2 + k^2 - 2kb_2 + b_2^2 + b_2^2 m_1^2 - 2b_2 m_1 m_2 (b_1 + k) + m_2^2 \left(b_1^2 + 2b_1 k + k^2\right) - \left((b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2\right) \\
&= 2kb_1 - 2b_1 b_2 + b_1^2 + k^2 - 2kb_2 + b_2^2 + b_2^2 m_1^2 - 2b_1 b_2 m_1 m_2 - 2kb_2 m_1 m_2 \\
&\quad + b_1^2 m_2^2 + 2kb_1 m_2^2 + k^2 m_2^2 - \left((b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2\right) \\
&= 2kb_1 - 2b_1 b_2 + b_1^2 + k^2 - 2kb_2 + b_2^2 + b_2^2 m_1^2 - 2b_1 b_2 m_1 m_2 - 2kb_2 m_1 m_2 + b_1^2 m_2^2 + 2kb_1 m_2^2 + k^2 m_2^2 - (b_1 - b_2)^2 - (m_1 b_2 - m_2 b_1)^2 \\
&= 2kb_1 - 2b_1 b_2 + b_1^2 + k^2 - 2kb_2 + b_2^2 + b_2^2 m_1^2 - 2b_1 b_2 m_1 m_2 - 2kb_2 m_1 m_2 \\
&\quad + b_1^2 m_2^2 + 2kb_1 m_2^2 + k^2 m_2^2 - b_1^2 + 2b_1 b_2 - b_2^2 - b_2^2 m_1^2 + 2b_1 b_2 m_1 m_2 - b_1^2 m_2^2 \\
&= 2kb_1 + 2kb_1 m_2^2 + k^2 - 2kb_2 + k^2 m_2^2 - 2kb_2 m_1 m_2
\end{aligned}$$

Recall, as discussed previously, that our goal was to find the difference in nudges between the LHS and the RHS; specifically, values in which the nudge of the LHS was less than that of the RHS. In this case, the RHS nudge impact is zero, so the inequality becomes:

$$\left(1 + m_2^2\right) k^2 + \left(2b_1 + 2b_1 m_2^2 - 2b_2 - 2b_2 m_1 m_2\right) k < 0$$

The similarity in structure to the nudge impact on slope:

$$\left(1 - b_2^2\right) j^2 + (2m_1 - 2b_2^2 m_1 - 2m_2 + 2b_1 b_2 m_2) j < 0$$

...is intriguing, and likely the result of the inequality being generally symmetric when all components are moved to one side.

Using the quadratic equation, we can solve for $k$ as:

$$k = \frac{-\left(2b_1 \pm 2b_1 m_2^2 - 2b_2 - 2b_2 m_1 m_2\right) \pm \sqrt{\left(2b_1 + 2b_1 m_2^2 - 2b_2 - 2b_2 m_1 m_2\right)^2 - 0}}{2\left(1 + m_2^2\right)}$$

Solving for one case:

$$\begin{aligned}
&\frac{-\left(2b_1 + 2b_1 m_2^2 - 2b_2 - 2b_2 m_1 m_2\right) + \sqrt{\left(2b_1 + 2b_1 m_2^2 - 2b_2 - 2b_2 m_1 m_2\right)^2 - 0}}{2\left(1 + m_2^2\right)} \\
&= \frac{-\left(2b_1 + 2m_2 b_1 - 2m_2 m_1 b_2 - 2b_2\right) + 2b_1 + 2m_2 b_1 - 2m_2 m_1 b_2 - 2b_2}{2\left(m_2 + 1\right)} \\
&= \frac{0}{2\left(1 + m_2\right)} \\
&= 0
\end{aligned}$$

Like the solutions for nudges to slope, one solution is 0.

$$
\begin{aligned}
&\frac{-\left(2b_1+2b_1m_2^2-2b_2-2b_2m_1m_2\right)-\sqrt{\left(2b_1+2b_1m_2^2-2b_2-2b_2m_1m_2\right)^2-0}}{2\left(1+m_2^2\right)}\\
&=\frac{-2\left(2b_1+2b_1m_2^2-2b_2-2b_2m_1m_2\right)}{2\left(1+m_2^2\right)}\\
&=-\frac{2b_1+2b_1m_2^2-2b_2-2b_2m_1m_2}{1+m_2^2}
\end{aligned}
$$

Another solution is $-\frac{2b_1+2b_1m_2^2-2b_2-2b_2m_1m_2}{1+m_2^2}$. Thus, solutions for $k$ are bounded by $(0, -\frac{2b_1+2b_1m_2^2-2b_2-2b_2m_1m_2}{1+m_2^2}$ (or could be rewritten as $-\frac{2b_1+2b_1m_2^2-2b_2-2b_2m_1m_2}{1+m_2^2}$). Hence, we can outline steps to alter lines to 'tip the inequality' to be true as follows:

1. Find the value of $-\frac{2b_1+2b_1m_2^2-2b_2-2b_2m_1m_2}{1+m_2^2}$.
2. If this value is larger than 0, increase $b_1$ by an infinitesimally small amount, $\frac{1}{a}$.
3. If this value is less than 0, decrease $b_1$ by an infinitesimally small amount, $\frac{1}{a}$.

We have established *three* methods of altering lines such that the inequality is satisfied by adding or subtracting infinitesimally small amounts to the slopes and $y$-intercepts.

## 3.4 Summary of Findings

In our exploration of minimum and maximum discrepancies - the primary focus of this paper - we have made a few findings. They are summarized below:

- The maximum possible discrepancy is formed at the origin.
  - Furthermore, the more 'polarized' (oppositely directed) two slopes are, the larger the discrepancy is.
  - This practice of 'rewarding' polarized slopes applies beyond lines going through the origin.
- The minimum possible discrepancy is formed by intersections on the circle or by lines whose slope varies by an infinitely small amount with the same $y$-intercept, regardless of intersection point (which would be the $y$-intercept).
  - An infinitely small discrepancy can be produced if the intersection is on the circle or if the intersection is on $y = 0$ (both $y$-intercepts are the same).
  - Intersections on the circle do not qualify as discrepancies, since they do not satisfy being *inside* the circle.
  - The following changes can be made to the slope of one of the two lines to form the minimum valid discrepancy:
    1. Find the value of $-\frac{2m_1-2m_2-2b_2^2m_1+2b_1b_2m_2}{1-b_2^2}$.
    2. If that value is larger than 0, make a positive infinitesimally small change, $\frac{1}{a}$, to $m_1$.
    3. If that value is less than 0, make a negative infinitesimally small change, $-\frac{1}{a}$, to $m_1$.
  - The following changes can be made to the $y$-intercept of one of the two lines to form the minimum valid discrepancy:
    1. Find the value of $-\frac{2b_1-2b_2+2b_1m_2^2-2b_2m_1m_2}{1+m_2^2}$.
    2. If this value is larger than 0, increase $b_1$ by an infinitesimally small amount, $\frac{1}{a}$.
    3. If this value is less than 0, decrease $b_1$ by an infinitesimally small amount, $\frac{1}{a}$.

# 4 Further Discussion

This environment houses many interesting relationships that can be further explored.

For values that fall between the maximum and minimum discrepancy of the LHS and the RHS, is the distance between the intersection of the point and the origin proportional to how large or small the discrepancy is? For instance, if the discrepancy between the LHS and RHS falls halfway in between the maximum and minimum values, is the point located halfway between intersections of lines with maximum and minimum discrepancies?

In this paper we explored the minimum discrepancy in terms of $a$ and found it to be $\frac{1}{a^2}$ for adjusting coincident lines. However, we did not do so for the Circle-Rim Intersection Conjecture. This was because finding the minimum possible value of the discrepancy $\text{RHS} - \text{LHS} = \left(\left(m_1 + \frac{1}{a}\right) - m_2\right)^2 - \left(\left(m_1 + \frac{1}{a}\right) b_2 - m_2 b_1\right)^2 - (b_1 - b_2)^2$ (the discrepancy produced when the slope was adjusted by an infinitely small amount) with not being able to cancel out any terms conveniently, like what was done with coincident lines, is daunting and not immediately obvious. A direction for further discussion would be to find the minimum possible discrepancies when one of the four maneuvers is changed by $\frac{1}{a}$. Furthermore, we did not exhaust all possible solutions to the three presented; it may be possible that there exist other methods of achieving minimum discrepancy outside of the ones discussed.

One notable application of these findings is to the probability problem, as described below.

## 4.1 The Probability Problem and Computational Results

The original motivation for exploring this topic was in finding the *probability* that two lines, with uniformly randomly selected values of $m_1$, $m_2$, $b_1$, and $b_2$ will intersect inside the circle. The answer, calculated by a program, converges to about 0.74995871 and remains so after only a few hundred iterations. This means that the intersection of any randomly chosen set of lines has a ≈74.99% chance of falling inside the unit circle.

```python
import numpy as np
def run_experiment(iter_num=100_000_000):
    counter = 0
    for i in range(iter_num):
        b1 = np.random.uniform(low=-1.0,high=1.0,size=None)
        b2 = np.random.uniform(low=-1.0,high=1.0,size=None)
        m1 = np.random.uniform(low=-2**31,high=2**31-1,size=None)
        m2 = np.random.uniform(low=-2**31,high=2**31-1,size=None)
        if (b1-b2)**2 + (m1*b2 - m2*b1)**2 < (m1-m2)**2: counter += 1
    return counter/iter_num
'''
Assumptions are made about the low and high values; it is placed at -2**31 and 2**31-1, respectively.
    Moving the lower and upper bounds much farther or much closer from/to 0 has very little effect on
    the actual calculated value, which remains within 0.01 of 0.74. Thus we assume it the result to
    generally be equivalent to if the lower and upper bounds are negative and positive infinity.
'''
```

Listing 1: Program used to calculate value.

This probability's closeness to $\frac{3}{4}$ is definitely intriguing. Can this be explained, or even derived, by considering the extreme cases and dynamics of the LHS and RHS of the original equation?

## 4.2 A Proposal for Approaching the Probability Problem

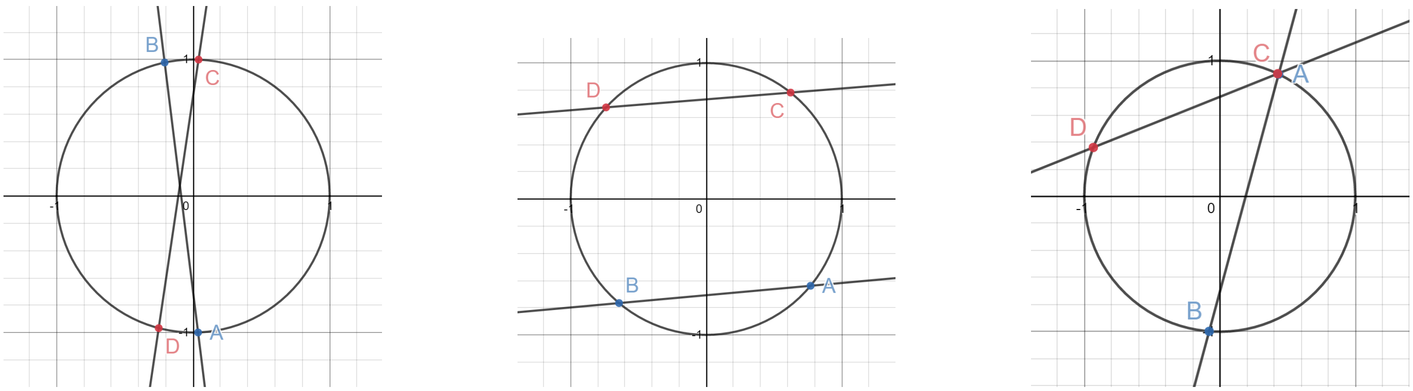
One laborious and difficult - given the current set of mathematical tools possessed - method of approaching this probability problem would be to take the inequality $(b_1 - b_2)^2 + (m_1 b_2 - m_2 b_1)^2 < (m_1 - m_2)^2$, find probability density functions, and use integrals to derive the solution.

Perhaps we can utilize another quirk of this problem: let the four intersection points be labelled $A$, $B$, $C$, and $D$, where $A$ and $B$ are on the same line and $C$ and $D$ are one another. Let $d(P)$ return the degree measure, in interval $[0°, 360°)$, which returns the difference in angles of the angle formed by a line segment from point $P$ to the origin and the line segment from the origin to $(1, 0)$. Thus, a point falling in the third quadrant would have a degree measure somewhere in the interval $(180°, 270°)$. If we are to write out $d(P)$ for each point $P$ in $A, B, C, D$ from least to greatest, the points must alternate by line for the intersection to fall inside the unit circle. That is, some acceptable inequalities include:

- $d(A) < d(C) < d(B) < d(D)$

- $d(B) < d(C) < d(A) < d(D)$
- $d(C) < d(A) < d(D) < d(B)$

Visually, this phenomenon is demonstrated in Figure 7.




(a) Order: $d(C) < d(B) < d(D) < d(A)$. Intersection is within the circle.

(b) Order: $d(C) < d(D) < d(B) < d(A)$. Intersection is outside the circle.

(c) Order: $d(A) = d(C) < d(D) < d(B)$. The point lands inside the unit circle boundary.

Figure 7: Visual demonstration of labelling points and comparing them by degrees.

There are only $4 \times 3 \times 2 \times 1 = 24$ possible orientations of $d(A), d(B), d(C)$, and $d(D)$ (ignoring case (c) above, in which two points may be at the same location). Furthermore, many are 'duplicates' - for instance, there is no real meaningful different between $d(A) < d(C) < d(B) < d(D)$ and $d(B) < d(C) < d(A) < d(D)$. It may be possible to partition the possibilities into categories, find the individual probabilities of each, aggregate them.

Using degrees seems more suited towards this environment of bridging circles and lines. It is likely that, as degrees are a closed system that better at modelling the dis-proportionality and bounds of slopes. Such an approach would incorporate findings discussed earlier about bounds.

## 4.3 The Bertrand Paradox and Why This Problem is Hard

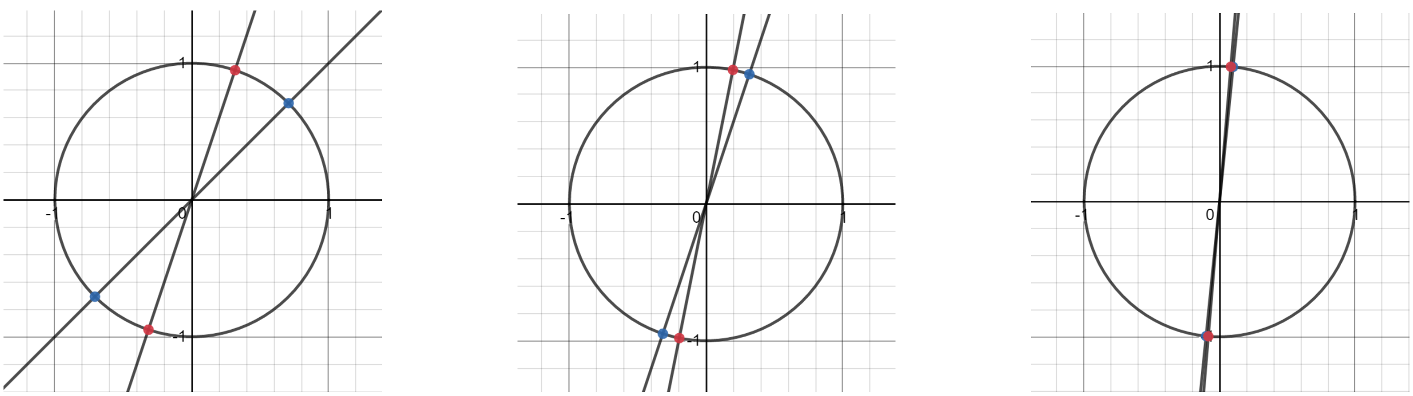
The Bertrand paradox discusses interpretations of randomness in probability theory. It describes the question:

> *Consider an equilateral triangle inscribed in a circle. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle?*

Given how 'random' chords of a circle are selected, Joseph Bertrand proved that three answers - $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{1}{4}$ - could all be correct. In the same spirit, this problem is hard because of the way 'randomness' is defined in relation to the question of intersecting chords. An increase in slope does not mean a proportional distance travelled in the intersection between the line and the circle. Consider the difference a slope of 2 makes depending on the 'objective' value of the slope, visualized in Figure 8.

Thus, what is difficult about imagining this problem conceptually is that with no bound on the slope, being $(-\infty, \infty)$, practically every line drawn should be almost vertical. Furthermore, our ideas of how random slopes and $y$-intercepts translates to the intersection of these two lines in an environment converts what should be clean and uniform into something puzzling.

Notably, the probability that any two randomly selected chords on a circle (by uniformly randomly selecting a degree) intersect is $\frac{1}{3}$. This is, however, not the case with respect to the given probability problem, because slopes and angles are not proportionate to each other. An interesting area to study would be in understanding how uniform randomness in the Cartesian-coordinate context can be 'projected' onto distorted randomness in the circular context.




(a) Order: Lines with slope 1 and 3.
(b) Lines with slope 3 and 5.
(c) Lines with slope 10 and 12.

Figure 8: Demonstrating the translation of a uniformly chosen slope to the warped surface of a circle.